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Periodic Solutions for Non-Linear Systems of Integral Equations

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Abstract: *The aim of this paper is to study the existence and approximation of periodic solutions for non-linear systems of integral equations, by using the numerical-analytic method which were introduced by Samoilenko[10, 11]. The study of such nonlinear integral equations is more general and leads us to improve and extend the results of Butris [2].*
Keyword And Phrases: *Numerical-analytic methods, existence and approximation of periodic solutions, nonlinear system, integral equations.*

I. Introduction

Integral equation has been arisen in many mathematical and engineering field, so that solving this kind of problems are more efficient and useful in many research branches. Analytical solution of this kind of equation is not accessible in general form of equation and we can only get an exact solution only in special cases. But in industrial problems we have not spatial cases so that we try to solve this kind of equations numerically in general format. Many numerical schemes are employed to give an approximate solution with sufficient accuracy [3,4,6, ,12,13,14,15]. Many author create and develop numerical-analytic methods [1, 2,5, 7,8,9] and schemes to investigate periodic solution of integral equations describing many applications in mathematical and engineering field.

Consider the following system of non-linear integral equations which has the form:

$$x(t,x_0,y_0) = F_0(t) + \int_0^t f_1(s, x(s,x_0,y_0), y(s,x_0,y_0), \int_{-\infty}^{s} G_1(s,\tau) g_1(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0))\, d\tau, \int_{a(s)}^{b(s)} g_1(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0)) d\tau) ds \quad \cdots (\mathrm{I}_1)$$

$$y(t,x_0,y_0) = G_0(t) + \int_0^t f_2(s, x(s,x_0,y_0), y(s,x_0,y_0), \int_{-\infty}^{s} G_2(s,\tau) g_2(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0))\, d\tau, \int_{a(s)}^{b(s)} g_2(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0)) d\tau) ds \quad \cdots (\mathrm{I}_2)$$

where $x \in D \subset R^n$, D is closed and bounded domain subset of Euclidean space R^n.

Let the vector functions
$f_1(t,x,y,z,w) = \big(f_{11}(t,x,y,z,w), f_{12}(t,x,y,z,w), \dots, f_{1n}(t,x,y,z,w)\big)$,
$f_2(t,x,y,u,v) = \big(f_{21}(t,x,y,u,v), f_{22}(t,x,y,u,v), \dots, f_{2n}(t,x,y,u,v)\big)$,
$g_1(t,x,y) = \big(g_{11}(t,x,y), g_{12}(t,x,y), \dots, g_{1n}(t,x,y)\big)$,
$g_2(t,x,y) = \big(g_{21}(t,x,y), g_{22}(t,x,y), \dots, g_{2n}(t,x,y)\big)$,
$F_0(t) = (F_{01}(t), F_{02}(t), \dots, F_{0n}(t))$,
and
$G_0(t) = (G_{01}(t), G_{02}(t), \cdots, G_{0n}(t))$
are defined and continuous on the domains:

$$\left.\begin{aligned}(t,x,y,z,w) \in R^1 \times D \times D_1 \times D_2 = (-\infty,\infty) \times D \times D_1 \times D_2 \times D_3\\ (t,x,y,u,v) \in R^1 \times D \times D_1 \times D_2 = (-\infty,\infty) \times D \times D_1 \times D_2 \times D_3\end{aligned}\right\} \cdots (1.1)$$

and periodic in t of period T, Also a(t) and b(t) are continuous and periodic in t of period T.

Suppose that the functions $f_1(t,x,y,z,w)$, $f_2(t,x,y,u,v)$, $g_1(t,x,y)$ and $g_2(t,x,y)$ satisfies the following inequalities:

$$\left.\begin{array}{ll}\|f_1(t,x,y,z,w)\| \le M_1\,, \|f_2(t,x,y,u,v)\| \le M_2\,, & M_1, M_2 > 0\\ \|g_1(t,x,y)\| \le N_1\,, \quad \|g_1(t,x,y)\| \le N_1\,, & N_1, N_2 > 0\end{array}\right\} \cdots (1.2)$$

$$\|f_1(t,x_1,y_1,z_1,w_1) - f_1(t,x_2,y_2,z_2,w_2)\| \le K_1\|x_1-x_2\| + K_2\|y_1-y_2\| + +K_3\|z_1-z_2\| + K_4\|w_1-w_2\| \quad \cdots (1.3)$$

$$\|f_2(t,x_1,y_1,u_1,v_1) - f_2(t,x_2,y_2,u_2,v_2)\| \le L_1\|x_1-x_2\| + L_2\|y_1-y_2\| + +L_3\|u_1-u_2\| + L_4\|v_1-v_2\|\,, \quad \cdots (1.4)$$

$$\|g_1(t,x_1,y_1) - g_1(t,x_2,y_2)\| \le R_1\|x_1-x_2\| + R_2\|y_1-y_2\|\,, \quad \cdots (1.5)$$

$$\|g_2(t,x_1,y_1) - g_2(t,x_2,y_2)\| \le H_1\|x_1-x_2\| + H_2\|y_1-y_2\|\,, \quad \cdots (1.6)$$

for all $t \in R^1$, $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1, z, z_1, z_2, u, u_1, u_2 \in D_2$, and $w, w_1, w_2, v, v_1, v_2 \in D_3$, where $M_1 = (M_{11}, M_{12}, \dots, M_{1n})$,
$M_2 = (M_{21}, M_{22}, \dots, M_{2n})$, $N_1 = (N_{11}, N_{12}, \dots, N_{1n})$ and
$N_2 = (N_{21}, N_{22}, \dots, N_{2n})$ are positive constant vectors and $K_1 = (K_{1_{ij}})$, $K_2 = (K_{2_{ij}})$, $K_3 = (K_{3_{ij}})$, $K_4 = (K_{4_{ij}})$, $L_1 = (L_{1_{ij}})$, $L_2 = (L_{2_{ij}})$,
$L_3 = (L_{3_{ij}})$, $L_4 = (L_{4_{ij}})$, $R_1 = (R_{1_{ij}})$, $R_2 = (R_{2_{ij}})$, $H_1 = (H_{1_{ij}})$, and
$H_2 = (H_{2_{ij}})$ are positive constant matrices.

Also $G_1(t,s)$ and $G_2(t,s)$ are (n×n) continuous positive matrix and periodic in t, s of period T in the domain $R^1 \times R^1$ and satisfy the following conditions:

$$\|G_1(t,s)\| \le \gamma\, e^{-\lambda_1(t-s)}\,, \quad \gamma,\ \lambda_1 > 0 \quad \cdots (1.7)$$

and

$$\|G_2(t,s)\| \le \delta\, e^{-\lambda_2(t-s)}\,, \quad \delta,\ \lambda_2 > 0 \quad \cdots (1.8)$$

where $-\infty < 0 \le s \le t \le T < \infty$, $i,j = 1,2,\cdots,n$,
and $h = max_{t\in[0,T]}|b(t) - a(t)|$, $\|.\| = max_{t\in[0,T]}|.|$

Now define a non-empty sets as follows:

$$\left.\begin{array}{l} D_{f_1} = D - \frac{T}{2}M_1\,,\\ D_{1f_2} = D_1 - \frac{T}{2}M_2\,,\\ D_{2f_1} = D_2 - \frac{T}{2}\left(\frac{\gamma}{\lambda_1}\right)(R_1M_1 + R_2M_2)\,,\\ D_{3f_1} = D_3 - \frac{T}{2}h(R_1M_1 + R_2M_2)\,,\\ D_{2f_2} = D_2 - \frac{T}{2}\left(\frac{\delta}{\lambda_2}\right)(H_1M_1 + H_2M_2)\,,\\ D_{3f_2} = D_3 - \frac{T}{2}h(H_1M_1 + H_2M_2)\,. \end{array}\right\} \cdots (1.9)$$

Forever, we suppose that the greatest eigen-value q_{max} of the following matrix:

$$Q_0 = \begin{pmatrix} \frac{T}{2}[K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)] & \frac{T}{2}[K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)] \\ \frac{T}{2}[L_1 + H_1(L_3\frac{\delta}{\lambda_2} + hL_4)] & \frac{T}{2}[L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)] \end{pmatrix} \cdots (1.10)$$

is less than unity, i.e.

$$q_{max}(Q_0) = \frac{\varphi_1 + \sqrt{\varphi_1{}^2 + 4(\varphi_2 - \varphi_3)}}{2} < 1\,, \quad \cdots (1.11)$$

where $\varphi_1 = \frac{T}{2}[K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)] + \frac{T}{2}[L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)]$,
$\varphi_2 = (\frac{T}{2}[K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)])(\frac{T}{2}[L_1 + H_1(\frac{\delta}{\lambda_2}L_3 + hL_4)])$ and
$\varphi_3 = (\frac{T}{2}[K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)])(\frac{T}{2}[L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)])$.

By using lemma 3.1[10],we can state and prove the following lemma:

Lemma 1.1. Let $f_1(t,x,y,z,w)$ and $f_2(t,x,y,u,v)$ be a vector functions which are defined and continuous in the interval $[0,T]$, then the inequality:

$$\begin{pmatrix}\|L_1(t,x_0,y_0)\| \\ \|L_2(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}M_1\alpha(t) \\ M_2\alpha(t)\end{pmatrix} \qquad \cdots (1.12)$$

satisfies for $0 \le t \le T$ and $\alpha(t) \le \frac{T}{2}$,

where $\alpha(t) = 2t(1-\frac{t}{T})$ for all $t \in [0,T]$,

$$L_1(t,x_0,y_0) = \int_0^t [f_1(s,x_0,y_0, \int_{-\infty}^{s} G_1(s,\tau)g_1(\tau,x_0,y_0)\,d\tau, , \int_{a(s)}^{b(s)} g_1(\tau,x_0,y_0)d\tau) - \frac{1}{T}\int_0^T f_1(s,x_0,y_0, , \int_{-\infty}^{s} G_1(s,\tau)g_1(\tau,x_0,y_0)\,d\tau, \int_{a(s)}^{b(s)} g_1(\tau,x_0,y_0)d\tau)ds]ds$$

$$L_2(t,x_0,y_0) = \int_0^t [f_2(s,x_0,y_0, \int_{-\infty}^{s} G_2(s,\tau)g_2(\tau,x_0,y_0)\,d\tau, , \int_{a(s)}^{b(s)} g_2(\tau,x_0,y_0)d\tau) - \frac{1}{T}\int_0^T f_2(s,x_0,y_0, , \int_{-\infty}^{s} G_2(s,\tau)g_2(\tau,x_0,y_0)\,d\tau, \int_{a(s)}^{b(s)} g_2(\tau,x_0,y_0)d\tau)ds]ds$$

Proof.

$$\|L_1(t,x_0,y_0)\| \le (1-\frac{t}{T})\int_0^t \left\| f_1(s,x_0,y_0, \int_{-\infty}^{s} G_1\,(s,\tau)g_1(\tau,x_0,y_0)d\tau, , \int_{a(s)}^{b(s)} g_1(\tau,x_0,y_0)d\tau)\right\| ds +$$

$$+\frac{t}{T}\int_t^T \left\| f_1(s,x_0,y_0, \int_{-\infty}^{s} G_1\,(s,\tau)g_1(\tau,x_0,y_0)d\tau, \int_{a(s)}^{b(s)} g_1(\tau,x_0,y_0)d\tau)\right\| ds$$

$$\le (1-\frac{t}{T})\int_0^t M_1\,ds + \frac{t}{T}\int_t^T M_1\,ds$$

$$\le M_1[(1-\frac{t}{T})t + \frac{t}{T}(T-t)]$$

so that:

$$\|L_1(t,x_0,y_0)\| \le M_1\alpha(t) \qquad \cdots (1.13)$$

And similarly, we get also

$$\|L_2(t,x_0,y_0)\| \le M_2\alpha(t) \qquad \cdots (1.14)$$

From (1.13) and (1.14), then the inequality (1.12) satisfies for all $0 \le t \le T$ and $\alpha(t) \le \frac{T}{2}$. ■

II. Approximation Solution Of (I_1) And (I_2)

In this section, we study the approximate periodic solution of (I_1) and (I_2) by proving the following theorem:

Theorem 2.1. If the system (I_1) and (I_2) satisfies the inequalities (1.2), (1.3), 1.4), (1.5),(1.6) and conditions(1.7),(1.8) has periodic solutions $x = x(t,x_0,y_0)$ and $y = y(t,x_0,y_0)$, then the sequence of functions:

$$x_{m+1}(t,x_0,y_0) = F_0(t) + \int_0^t [f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),$$

$$, \int_{-\infty}^{s} G_1(s,\tau) g_1(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_1(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0)) d\tau) - \frac{1}{T}\int_0^T f_1(s, x_m(s, x_0, y_0),$$

$$, y_m(s, x_0, y_0), \int_{-\infty}^{s} G_1(s,\tau) g_1(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_1(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0)) d\tau) ds] ds \qquad \cdots (2.1)$$

with

$x_0(t, x_0, y_0) = F_0(t) = x_0$, for all m $= 0, 1, 2, \cdots$

$$y_{m+1}(t, x_0, y_0) = G_0(t) + \int_0^t [f_2(s, x_m(s, x_0, y_0), y_m(s, x_0, y_0),$$

$$, \int_{-\infty}^{s} G_2(s,\tau) g_2(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_2(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0)) d\tau) - \frac{1}{T}\int_0^T f_2(s, x_m(s, x_0, y_0),$$

$$, y_m(s, x_0, y_0), \int_{-\infty}^{s} G_2(s,\tau) g_2(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_2(\tau, x_m(\tau, x_0, y_0), y_m(\tau, x_0, y_0)) d\tau) ds] ds \qquad \cdots (2.2)$$

with

$y_0(t, x_0, y_0) = G_0(t) = y_0$, for all m $= 0, 1, 2, \cdots$

periodic in t of period T, and uniformly converges as $m \to \infty$ in the domain:

$$(t, x_0, y_0) \in [0, T] \times D_{f_1} \times D_{1f_2} \qquad \cdots (2.3)$$

to the limit functions $x^0(t, x_0, y_0)$, $y^0(t, x_0, y_0)$ defined in the domain (2.3) which are periodic in t of period T and satisfying the system of integral equations:

$$x(t, x_0, y_0) = F_0(t) + \int_0^t [f_1(s, x(s, x_0, y_0), y(s, x_0, y_0),$$

$$, \int_{-\infty}^{s} G_1(s,\tau) g_1(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_1(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0)) d\tau) - \frac{1}{T}\int_0^T f_1(s, x(s, x_0, y_0),$$

$$, y(s, x_0, y_0), \int_{-\infty}^{s} G_1(s,\tau) g_1(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_1(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0)) d\tau) ds] ds \qquad \cdots (2.4)$$

and

$$y(t, x_0, y_0) = G_0(t) + \int_0^t [f_2(s, x(s, x_0, y_0), y(s, x_0, y_0),$$

$$, \int_{-\infty}^{s} G_2(s,\tau) g_2(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_2(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0)) d\tau) - \frac{1}{T}\int_0^T f_2(s, x(s,x_0,y_0),$$

$$, y(s,x_0,y_0), \int_{-\infty}^{s} G_2(s,\tau) g_2(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_2(\tau, x(\tau,x_0,y_0), y(\tau,x_0,y_0)) d\tau) ds] ds \qquad \cdots (2.5)$$

provided that:

$$\begin{pmatrix} \|x^0(t,x_0,y_0) - x_m(t,x_0,y_0)\| \\ \|y^0(t,x_0,y_0) - y_m(t,x_0,y_0)\| \end{pmatrix} \le {Q_0}^m (E - Q_0)^{-1} C_0 \qquad \cdots (2.6)$$

for all $m \ge 0$, where

$C_0 = \begin{pmatrix} \frac{T}{2} M_1 \\ \frac{T}{2} M_2 \end{pmatrix}$, E is identity matrix.

Proof. Consider the sequence of functions $x_1(t,x_0,y_0), x_2(t,x_0,y_0), \cdots,$ $x_m(t,x_0,y_0), \cdots$, and $y_1(t,x_0,y_0), y_2(t,x_0,y_0), \cdots, y_m(t,x_0,y_0), \cdots$, defined by the recurrences relations (2.1) and (2.2). Each of these functions of sequence defined and continuous in the domain (1.1) and periodic in t of period T.

Now, by the lemma 1.1, and using the sequence of functions (2.1), when $m = 0$, we get:

By mathematical induction and lemma1.1 , we have the following inequality:

$$\|x_m(t,x_0,y_0) - x_0\| \le \frac{T}{2} M_1 \qquad \cdots (2.13)$$

i.e. $x_m(t,x_0,y_0) \in D$, for all $t \in R^1, x_0 \in D_{f_1}, y_0 \in D_{1f_2}$, $m = 0,1,2,\cdots,$

and

$$\|y_m(t,x_0,y_0) - y_0\| \le \frac{T}{2} M_2 \qquad \cdots (2.14)$$

i.e. $y_m(t,x_0,y_0) \in D$, for all $t \in R^1, x_0 \in D_{f_1}, y_0 \in D_{1f_2}, m = 0,1,2,\cdots,$

Now, from (2.13), we get:

$$\|z_m(t,x_0,y_0) - z_0(t,x_0,y_0)\| \le \frac{T}{2}(\frac{\gamma}{\lambda_1})(R_1 M_1 + R_2 M_2) \qquad \cdots (2.15)$$

and

$$\|w_m(t,x_0,y_0) - w_0(t,x_0,y_0)\| \le \frac{T}{2} h (R_1 M_1 + R_2 M_2) \qquad \cdots (2.16)$$

for all $t \in R^1, x_0 \in D_{f_1}, y_0 \in D_{1f_2}$, $z_0(t,x_0,y_0) \in D_{2f_1}$ and $w_0(t,x_0,y_0) \in D_{3f_1}$,

i.e. $z_m(t,x_0,y_0) \in D_2$ and $w_m(t,x_0,y_0) \in D_3$, for all $x_0 \in D_{f_1}, y_0 \in D_{1f_2}$,

where

$$z_m(t,x_0,y_0) = \int_{-\infty}^{t} G_1(t,s) g_1(s, x_m(s,x_0,y_0), y_m(s,x_0,y_0)) ds$$

and

$$w_m(t,x_0,y_0) = \int_{a(t)}^{b(t)} g_1(s, x_m(s,x_0,y_0), y_m(s,x_0,y_0)) ds, \qquad m = 0,1,2,\cdots,$$

Also from (4.2.14), we get:

$$\|u_m(t,x_0,y_0) - u_0(t,x_0,y_0)\| \le \frac{T}{2}(\frac{\delta}{\lambda_2})(H_1 M_1 + H_2 M_2) \qquad \cdots (2.17)$$

and

$$\|V_m(t,x_0,y_0) - V_0(t,x_0,y_0)\| \le \frac{T}{2} h (H_1 M_1 + H_2 M_2) \qquad \cdots (2.18)$$

for all $t \in R^1, x_0 \in D_{f_1}, y_0 \in D_{1f_2}$, $u_0(t, x_0, y_0) \in D_{2f_2}$ and
$v_0(t, x_0, y_0) \in D_{3f_2}$,
i.e. $u_m(t, x_0, y_0) \in D_2$ and $v_m(t, x_0, y_0) \in D_3$, for all $x_0 \in D_{f_1}, y_0 \in D_{1f_2}$,
where

$$u_m(t, x_0, y_0) = \int_{-\infty}^{t} G_2(t, s) g_2\big(s, x_m(s, x_0, y_0), y_m(s, x_0, y_0)\big) ds$$

and

$$v_m(t, x_0, y_0) = \int_{a(t)}^{b(t)} g_2\big(s, x_m(s, x_0, y_0), y_m(s, x_0, y_0)\big) ds, \qquad m = 0,1,2, \cdots,$$

Next, we claim that two sequences $\{x_m(t, x_0, y_0)\}_{m=0}^{\infty}, \{y_m(t, x_0, y_0)\}_{m=0}^{\infty}$ are convergent uniformly to the limit functions x, y on the domain (2.3).

By mathematical induction and lemma 1.1, we find that:

$$\begin{aligned}\|x_{m+1}(t, x_0, y_0) - x_m(t, x_0, y_0)\| \leq \\ \leq \alpha(t)[K_1 + R_1(\frac{\gamma}{\lambda_1} K_3 + hK_4)] \|x_m(t, x_0, y_0) - x_{m-1}(t, x_0, y_0)\| \\ + \alpha(t)[K_2 + R_2(\frac{\gamma}{\lambda_1} K_3 + hK_4)] \|y_m(t, x_0, y_0) - y_{m-1}(t, x_0, y_0)\|\end{aligned} \qquad \cdots (2.19)$$

and

$$\begin{aligned}\|y_{m+1}(t, x_0, y_0) - y_m(t, x_0, y_0)\| \leq \\ \leq \alpha(t)[L_1 + H_1(\frac{\delta}{\lambda_2} L_3 + hL_4)] \|x_m(t, x_0, y_0) - x_{m-1}(t, x_0, y_0)\| + \\ + \alpha(t)[L_2 + H_2(\frac{\delta}{\lambda_2} L_3 + hL_4)] \|y_m(t, x_0, y_0) - y_{m-1}(t, x_0, y_0)\|\end{aligned} \qquad \cdots (2.20)$$

We can write the inequalities (2.19) and (2.20) in a vector form:

$$C_{m+1}(t, x_0, y_0) \leq Q(t) C_m(t, x_0, y_0) \qquad \cdots (2.21)$$

where

$$C_{m+1}(t, x_0, y_0) = \begin{pmatrix} \|x_{m+1}(t, x_0, y_0) - x_m(t, x_0, y_0)\| \\ \|y_{m+1}(t, x_0, y_0) - y_m(t, x_0, y_0)\| \end{pmatrix},$$

and

$$Q(t) = \begin{pmatrix} \alpha(t)[K_1 + R_1(\frac{\gamma}{\lambda_1} K_3 + hK_4)] & \alpha(t)[K_2 + R_2(\frac{\gamma}{\lambda_1} K_3 + hK_4)] \\ \alpha(t)[L_1 + H_1(\frac{\delta}{\lambda_2} L_3 + hL_4)] & \alpha(t)[L_2 + H_2(\frac{\delta}{\lambda_2} L_3 + hL_4)] \end{pmatrix},$$

$$C_m(t, x_0, y_0) = \begin{pmatrix} \|x_m(t, x_0, y_0) - x_{m-1}(t, x_0, y_0)\| \\ \|y_m(t, x_0, y_0) - y_{m-1}(t, x_0, y_0)\| \end{pmatrix}$$

Now, we take the maximum value to both sides of the inequality (2.21), for all $0 \leq t \leq T$ and $\alpha(t) \leq \frac{T}{2}$, we get:

$$C_{m+1} \leq Q_0 C_m \quad , \qquad \cdots (2.22)$$

where $Q_0 = max_{t \in [0,T]} |Q(t)|$.

$$Q_0 = \begin{pmatrix} \frac{T}{2}[K_1 + R_1(\frac{\gamma}{\lambda_1} K_3 + hK_4)] & \frac{T}{2}[K_2 + R_2(\frac{\gamma}{\lambda_1} K_3 + hK_4)] \\ \frac{T}{2}[L_1 + H_1(\frac{\delta}{\lambda_2} L_3 + hL_4)] & \frac{T}{2}[L_2 + H_2(\frac{\delta}{\lambda_2} L_3 + hL_4)] \end{pmatrix}$$

By iterating the inequality (4.2.22), we find that:

$$C_{m+1} \leq {Q_0}^m C_0 \quad , \qquad \cdots (2.23)$$

which leads to the estimate:

$$\sum_{i=1}^{m} C_i \leq \sum_{i=1}^{m} {Q_0}^{i-1} C_0 \qquad \cdots (2.24)$$

Since the matrix Q_0 has maximum eigen-values of (4.1.13) and the series (4.2.24) is uniformly convergent, i. e.

$$\lim_{m\to\infty}\sum_{i=1}^{m} {Q_0}^{i-1}\, C_0 = \sum_{i=1}^{\infty} {Q_0}^{i-1}\, C_0 = (E-Q_0)^{-1}C_0 \qquad \cdots(2.25)$$

The limiting relation (4.2.25) signifies a uniform convergence of the sequence $\{x_m(t,x_0,y_0), y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ in the domain (4.2.3) as $m\to\infty$.
Let

$$\left.\begin{aligned} \lim_{m\to\infty} x_m\,(t,x_0,y_0) &= x^0(t,x_0,y_0)\ ,\\ \lim_{m\to\infty} y_m\,(t,x_0,y_0) &= y^0(t,x_0,y_0)\ . \end{aligned}\right\} \qquad \cdots(2.26)$$

Finally, we show that $x(t,x_0,y_0)\equiv x^0(t,x_0,y_0)\in D$ and $y(t,x_0,y_0)\equiv y^0(t,x_0,y_0)\in D_1$, for all $x_0\in D_{f_1}$ and $y_0\in D_{1f_2}$.
By using inequalities (2.1) and (2.4) and lemma 1.1, such that:

$$\Bigg\|\int_0^t [f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0)) -$$
$$-\frac{1}{T}\int_0^T f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0))ds]ds -$$
$$-\int_0^t [f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0)) -$$
$$-\frac{1}{T}\int_0^T f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0))ds]ds\Bigg\|$$
$$\le \alpha(t)\ \Big([K_1+R_1(\frac{\gamma}{\lambda_1}K_3+hK_4)]\|x_m(s,x_0,y_0)-x(s,x_0,y_0)\| +$$
$$+[K_2+R_2(\tfrac{\gamma}{\lambda_1}K_3+hK_4)]\|y_m(s,x_0,y_0)-y(s,x_0,y_0)\|\Big)\le$$
$$\le\frac{T}{2}\Big([K_1+R_1(\frac{\gamma}{\lambda_1}K_3+hK_4)]\|x_m(t,x_0,y_0)-x(t,x_0,y_0)\| +$$
$$+[K_2+R_2(\tfrac{\gamma}{\lambda_1}K_3+hK_4)]\|y_m(t,x_0,y_0)-y(t,x_0,y_0)\|\Big)\ .$$

From inequality (2.26) and on the other hand suppose that:.
$\|x_m(t,x_0,y_0)-x(t,x_0,y_0)\|\le\epsilon_1$,
$\|y_m(t,x_0,y_0)-y(t,x_0,y_0)\|\le\epsilon_2$.

Thus

$$\Bigg\|\int_0^t [f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0)) -$$
$$-\frac{1}{T}\int_0^T f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0))ds]ds -$$
$$-\int_0^t [f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0)) -$$
$$-\frac{1}{T}\int_0^T f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0))ds]ds\Bigg\|$$
$$\le\frac{T}{2}[K_1+R_1(\frac{\gamma}{\lambda_1}K_3+hK_4)]\epsilon_1+\frac{T}{2}[K_2+R_2(\frac{\gamma}{\lambda_1}K_3+hK_4)]\epsilon_2$$

Putting $\epsilon_1=\frac{\epsilon_3}{\frac{T}{2}[K_1+R_1(\frac{\gamma}{\lambda_1}K_3+hK_4)]}$ and $\epsilon_2=\frac{\epsilon_4}{\frac{T}{2}[K_2+R_2(\frac{\gamma}{\lambda_1}K_3+hK_4)]}$ and substituting in the last equation, we have:

$$\Bigg\|\int_0^t [f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0)) -$$

$$-\frac{1}{T}\int_0^T f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0))ds]ds -$$

$$-\int_0^t [f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0)) -$$

$$-\frac{1}{T}\int_0^T f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0))ds]ds\Bigg\|$$

$$\le \frac{T}{2}[K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\frac{\epsilon_3}{\frac{T}{2}[K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)]} +$$

$$+\frac{T}{2}[K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\frac{\epsilon_4}{\frac{T}{2}[K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]}$$

$$\le \epsilon_3 + \epsilon_{4,}$$

and choosing $\epsilon_3 + \epsilon_4 = \epsilon$, we get:

$$\Bigg\|\int_0^t [f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0)) -$$

$$-\frac{1}{T}\int_0^T f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0))ds]ds -$$

$$-\int_0^t [f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0)) -$$

$$-\frac{1}{T}\int_0^T f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0))ds]ds\Bigg\| \le \epsilon$$

for all $m \ge 0$,

$$\text{i.e.} \lim_{m\to\infty}\int_0^t [f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0)) -$$

$$-\frac{1}{T}\int_0^T f_1(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),z_m(s,x_0,y_0),w_m(s,x_0,y_0))ds]ds =$$

$$-\int_0^t [f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0)) -$$

$$-\frac{1}{T}\int_0^T f_1(s,x(s,x_0,y_0),y(s,x_0,y_0),z(s,x_0,y_0),w(s,x_0,y_0))ds]ds.$$

So $x(t,x_0,y_0) \in D$, and $x(t,x_0,y_0) \equiv x^0(t,x_0,y_0)$ is a periodic solution of (I1),

Also by using the same method above we can prove $y(t,x_0,y_0) \in D_1$, and $y(t,x_0,y_0) \equiv y^0(t,x_0,y_0)$ is also periodic solution of (I_2).

Theorem 2.2. With the hypotheses and all conditions of the theorem 2.1, the periodic solution of integral equations (I_1) and (I_2) are a unique on the domain (1.3).

Proof. Suppose that $\hat{x}(t,x_0,y_0)$ and $\hat{y}(t,x_0,y_0)$ be another periodic solutions for the systems (I_1) and (I_2) defined and continuous and periodic in t of period T, this means that:

$$\hat{x}(t,x_0,y_0) = F_0(t) + \int_0^t [f_1(s,\hat{x}(s,x_0,y_0),\hat{y}(s,x_0,y_0),$$

$$,\int_{-\infty}^s G_1(s,\tau)g_1(\tau,\hat{x}(\tau,x_0,y_0),\hat{y}(\tau,x_0,y_0))\,d\tau,$$

$$, \int_{a(s)}^{b(s)} g_1(\tau, \hat{x}(\tau, x_0, y_0), \hat{y}(\tau, x_0, y_0))d\tau) - \frac{1}{T}\int_0^T f_1(s, \hat{x}(s, x_0, y_0),$$

$$, \hat{y}(s, x_0, y_0), \int_{-\infty}^{s} G_1(s,\tau) g_1(\tau, \hat{x}(\tau, x_0, y_0), \hat{y}(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_1(\tau, \hat{x}(\tau, x_0, y_0), \hat{y}(\tau, x_0, y_0))d\tau) ds]ds \qquad \cdots (2.27)$$

and

$$\hat{y}(t, x_0, y_0) = G_0(t) + \int_0^t [f_2(s, \hat{x}(s, x_0, y_0), \hat{y}(s, x_0, y_0),$$

$$, \int_{-\infty}^{s} G_2(s,\tau) g_2(\tau, \hat{x}(\tau, x_0, y_0), \hat{y}(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_2(\tau, \hat{x}(\tau, x_0, y_0), \hat{y}(\tau, x_0, y_0))d\tau) - \frac{1}{T}\int_0^T f_2(s, \hat{x}(s, x_0, y_0),$$

$$, \hat{y}(s, x_0, y_0), \int_{-\infty}^{s} G_2(s,\tau) g_2(\tau, \hat{x}(\tau, x_0, y_0), \hat{y}(\tau, x_0, y_0))\, d\tau,$$

$$, \int_{a(s)}^{b(s)} g_2(\tau, \hat{x}(\tau, x_0, y_0), \hat{y}(\tau, x_0, y_0))d\tau) ds]ds \qquad \cdots (2.28)$$

For their difference, we should obtain the inequality:

$$\|x(t, x_0, y_0) - \hat{x}(t, x_0, y_0)\| \le$$

$$\le \frac{T}{2}\, [K_1 + R_1(\frac{\gamma}{\lambda_1} K_3 + hK_4)]\|x(t, x_0, y_0) - \hat{x}(t, x_0, y_0)\| +$$

$$+\frac{T}{2}[K_2 + R_2(\frac{\gamma}{\lambda_1} K_3 + hK_4)]\|y(t, x_0, y_0) - \hat{y}(t, x_0, y_0)\| \cdots (2.29)$$

And also

$$\|y(s, x_0, y_0) - \hat{y}(s, x_0, y_0)\| \le$$

$$\le (1 - \frac{t}{T}) \int_0^t [L_1\|x(s, x_0, y_0) - \hat{x}(s, x_0, y_0)\| + L_2\|y(s, x_0, y_0) - \hat{y}(s, x_0, y_0)\|$$

$$+\frac{\delta}{\lambda_2} L_3 (H_1\|x(s, x_0, y_0) - \hat{x}(s, x_0, y_0)\| + H_2\|y(s, x_0, y_0) - \hat{y}(s, x_0, y_0)\|) +$$

$$+hL_4(H_1\|x(s, x_0, y_0) - \hat{x}(s, x_0, y_0)\| + H_2\|y(s, x_0, y_0) - \hat{y}(s, x_0, y_0)\|)]ds$$

$$+\frac{t}{T}\int_t^T [L_1\|x(s, x_0, y_0) - \hat{x}(s, x_0, y_0)\| + L_2\|y(s, x_0, y_0) - \hat{y}(s, x_0, y_0)\|) +$$

$$+\frac{\delta}{\lambda_2} L_3 (H_1\|x(s, x_0, y_0) - \hat{x}(s, x_0, y_0)\| + H_2\|y(s, x_0, y_0) - \hat{y}(s, x_0, y_0)\|) +$$

$$+hL_4(H_1\|x(s, x_0, y_0) - \hat{x}(s, x_0, y_0)\| + H_2\|y(s, x_0, y_0) - \hat{y}(s, x_0, y_0)\|)]ds$$

$$\le \alpha(t)[L_1 + H_1(\frac{\delta}{\lambda_2} L_3 + hL_4)]\|x(t, x_0, y_0) - \hat{x}(t, x_0, y_0)\| +$$

$$+\alpha(t)[L_2 + H_2(\frac{\delta}{\lambda_2} L_3 + hL_4)]\|y(t, x_0, y_0) - \hat{y}(t, x_0, y_0)\|$$

so that:

$$\|y(t, x_0, y_0) - \hat{y}(t, x_0, y_0)\| \le$$

$$\le \frac{T}{2}[L_1 + H_1(\frac{\delta}{\lambda_2} L_3 + hL_4)]\|x(t, x_0, y_0) - \hat{x}(t, x_0, y_0)\| +$$

$$+\frac{T}{2}[L_2 + H_2(\frac{\delta}{\lambda_2} L_3 + hL_4)]\|y(t, x_0, y_0) - \hat{y}(t, x_0, y_0)\| \cdots (2.30)$$

and the inequalities (4.2.29) and (4.2.30) would lead to the estimate:

$$\begin{pmatrix} \|x(t,x_0,y_0)-\hat{x}(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-\hat{y}(t,x_0,y_0)\| \end{pmatrix} \le Q_0 \begin{pmatrix} \|x(t,x_0,y_0)-\hat{x}(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-\hat{y}(t,x_0,y_0)\| \end{pmatrix} \quad \cdots (2.31)$$

By iterating the inequality (4.2.27), which should find:

$$\begin{pmatrix} \|x(t,x_0,y_0)-\hat{x}(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-\hat{y}(t,x_0,y_0)\| \end{pmatrix} \le {Q_0}^m \begin{pmatrix} \|x(t,x_0,y_0)-\hat{x}(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-\hat{y}(t,x_0,y_0)\| \end{pmatrix}.$$

But ${Q_0}^m \to 0$ as $m \to \infty$, so that proceeding in the last inequality which is contradict the supposition It follows immediately $x(t,x_0,y_0)=\hat{x}(t,x_0,y_0)$ and
$y(t,x_0,y_0)=\hat{y}(t,x_0,y_0)$.

III. Existence of Solution of (I_1) and (I_2)

The problem of existence of periodic solution of period T of the system
(I_1) and (I_2) are uniquely connected with the existence of zero of the functions $\Delta_1(0,x_0,y_0)=\Delta_1$ and $\Delta_2(0,x_0,y_0)=\Delta_2$ which has the form:

$\Delta_1 : D_{f_1} \times D_{1f_2} \to R^n$

$$\Delta_1(0,x_0,y_0) = \frac{1}{T}\int_0^T f_1\Big(t, x^0(t,x_0,y_0), y^0(t,x_0,y_0), \int_{-\infty}^{t} G_1(t,s) g_1\big(s,x^0(s,x_0,y_0),y^0(s,x_0,y_0)\big)\,ds, \int_{a(t)}^{b(t)} g_1\big(s,x^0(s,x_0,y_0),y^0(s,x_0,y_0)\big)ds\Big)dt \quad \cdots (3.1)$$

$\Delta_2 : D_{f_1} \times D_{1f_2} \to R^n$

$$\Delta_2(0,x_0,y_0) = \frac{1}{T}\int_0^T f_2\Big(t, x^0(t,x_0,y_0), y^0(t,x_0,y_0), \int_{-\infty}^{t} G_2(t,s) g_2\big(s,x^0(s,x_0,y_0),y^0(s,x_0,y_0)\big)\,ds, \int_{a(t)}^{b(t)} g_2\big(s,x^0(s,x_0,y_0),y^0(s,x_0,y_0)\big)ds\Big)dt \quad \cdots (3.2)$$

where the function $x^0(t,x_0,y_0)$ is the limit of the sequence of the functions $x_m(t,x_0,y_0)$ and the function $y^0(t,x_0,y_0)$ is the limit of the sequence of the functions $y_m(t,x_0,y_0)$.
Since this two functions are approximately determined from the sequences of functions:

$\Delta_{1m} : D_{f_1} \times D_{1f_2} \to R^n$

$$\Delta_{1m}(0,x_0,y_0) = \frac{1}{T}\int_0^T f_1\Big(t, x_m(t,x_0,y_0), y_m(t,x_0,y_0), \int_{-\infty}^{t} G_1(t,s) g_1\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\big)\,ds, \int_{a(t)}^{b(t)} g_1\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\big)ds\Big)dt \quad \cdots (3.3)$$

$\Delta_{2m} : D_{f_1} \times D_{1f_2} \to R^n$

$$\Delta_{2m}(0,x_0,y_0) = \frac{1}{T}\int_0^T f_2\Big(t, x_m(t,x_0,y_0), y_m(t,x_0,y_0), \int_{-\infty}^{t} G_2(t,s) g_2\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\big)\,ds$$

$$, \int_{a(t)}^{b(t)} g_2\big(s, x_m(s, x_0, y_0), y_m(s, x_0, y_0)\big)ds)dt \qquad \cdots (3.4)$$

for all $m = 0,1,2,\cdots$

Theorem 3.1. If the hypotheses and all conditions of the theorem 2.1 and 2.2 are satisfied, then the following inequality satisfied:

$$\|\Delta_1(0, x_0, y_0) - \Delta_{1m}(0, x_0, y_0)\| \le d_m \qquad \cdots (3.5)$$

$$\|\Delta_2(0, x_0, y_0) - \Delta_{2m}(0, x_0, y_0)\| \le \eta_m \qquad \cdots (3.6)$$

satisfied for all $m \ge 0$, $x_0 \in D_{f_1}$ and $y_0 \in D_{1f_2}$,

where

$$d_m = \langle\Big([K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)] \qquad [K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\Big), , {Q_0}^{m+1}(E - Q_0)^{-1}C_0\rangle$$

and

$$\eta_m = \langle\Big([L_1 + H_1(\frac{\delta}{\lambda_2}L_3 + hL_4)] \qquad [L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)]\Big), , {Q_0}^{m+1}(E - Q_0)^{-1}C_0\rangle.$$

Proof. By using the relation (3.1) and (3.3), we have:

$$\|\Delta_1(0, x_0, y_0) - \Delta_{1m}(0, x_0, y_0)\|$$
$$\le [K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\|x^0(t, x_0, y_0) - x_m(t, x_0, y_0)\| +$$
$$+[K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\|y^0(t, x_0, y_0) - y_m(t, x_0, y_0)\|$$
$$\le \langle\Big([K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)] \qquad [K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\Big), , {Q_0}^{m+1}(E - Q_0)^{-1}C_0\rangle = d_m$$

And also by using the relation (3.2) and (3.4), we get:

$$\|\Delta_2(0, x_0, y_0) - \Delta_{2m}(0, x_0, y_0)\| \le$$
$$\le \langle\Big([L_1 + H_1(\frac{\delta}{\lambda_2}L_3 + hL_4)] \qquad [L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)]\Big), , {Q_0}^{m+1}(E - Q_0)^{-1}C_0\rangle = \eta_m$$

where $\langle . \rangle$ denotes the ordinary scalar product in the space R^n. ■

Theorem 3.2. Let the vector functions $f_1(t, x, y, z, w), f_2(t, x, y, u, v)$, $g_1(t, x, y)$ and $g_2(t, x, y)$ be defined on the domain:
$G = \{0 \le s \le t \le T, a \le x, y \le b, c \le z, u \le d, e \le w, v \le f\} \subseteq R^1$,
and periodic in t of period T.

Assume that the sequence of functions (4.3.3) and (4.3.4) satisfies the inequalities:

$$\left.\begin{aligned} &\min_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{1m}(0, x_0, y_0) \le -d_m \ , \\ &\max_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{1m}(0, x_0, y_0) \ge d_m \ , \end{aligned}\right\} \qquad \cdots (3.7)$$

$$\left.\begin{aligned} &\min_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{2m}(0, x_0, y_0) \le -\eta_m \ , \\ &\max_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{2m}(0, x_0, y_0) \ge \eta_m \ , \end{aligned}\right\} \qquad \cdots (3.8)$$

for all $m \ge 0$, where

$$d_m = \langle\Big([K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)] \qquad [K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\Big), , {Q_0}^{m+1}(E - Q_0)^{-1}C_0\rangle$$

and

$\eta_m = \langle\Big([L_1 + H_1(\frac{\delta}{\lambda_2}L_3 + hL_4)] \qquad [L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)]\Big),$
$, Q_0{}^{m+1}(E - Q_0)^{-1}C_0\rangle$

Then the system (4.1.1) and (4.1.2) has periodic solution of period T
$x = x(t, x_0, y_0)$ and $y = y(t, x_0, y_0)$ for which $x_0 \in [a + \frac{T}{2}M_1, b - \frac{T}{2}M_1]$ and
$y_0 \in [c + \frac{T}{2}M_2, d - \frac{T}{2}M_2]$.

Proof. Let x_1, x_2 be any two points in the interval $[a + \frac{T}{2}M_1, b - \frac{T}{2}M_1]$ and y_1, y_2 be any two points in the interval $[c + \frac{T}{2}M_2, d - \frac{T}{2}M_2]$,
such that:

$$\left.\begin{aligned} \Delta_{1m}(0, x_1, y_1) &= \min_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{1m}(0, x_0, y_0) \quad , \\ \Delta_{1m}(0, x_2, y_2) &= \max_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{1m}(0, x_0, y_0) \quad , \end{aligned}\right\} \cdots (3.9)$$

$$\left.\begin{aligned} \Delta_{2m}(0, x_1, y_1) &= \min_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{2m}(0, x_0, y_0) \quad , \\ \Delta_{2m}(0, x_2, y_2) &= \max_{\substack{a+\frac{T}{2}M_1 \le x_0 \le b-\frac{T}{2}M_1 \\ c+\frac{T}{2}M_2 \le y_0 \le d-\frac{T}{2}M_2}} \Delta_{2m}(0, x_0, y_0) \quad , \end{aligned}\right\} \cdots (3.10)$$

By using the inequalities (3.5), (3.6), (3.7) and (3.8), we have:

$$\left.\begin{aligned} \Delta_1(0, x_1, y_1) &= \Delta_{1m}(0, x_1, y_1) + [\Delta_1(0, x_1, y_1) - \Delta_{1m}(0, x_1, y_1)] \le 0 \quad , \\ \Delta_1(0, x_2, y_2) &= \Delta_{1m}(0, x_2, y_2) + [\Delta_1(0, x_2, y_2) - \Delta_{1m}(0, x_2, y_2)] \le 0 \quad . \end{aligned}\right\} \cdots (3.11)$$

$$\left.\begin{aligned} \Delta_2(0, x_1, y_1) &= \Delta_{2m}(0, x_1, y_1) + [\Delta_2(0, x_1, y_1) - \Delta_{2m}(0, x_1, y_1)] \le 0 \quad , \\ \Delta_2(0, x_2, y_2) &= \Delta_{2m}(0, x_2, y_2) + [\Delta_2(0, x_2, y_2) - \Delta_{2m}(0, x_2, y_2)] \le 0 \quad . \end{aligned}\right\} \cdots (3.12)$$

It follows from the inequalities (3.11) and (3.12) in virtue of the continuity of the functions $\Delta_1(0, x_0, y_0)$ and $\Delta_1(0, x_0, y_0)$ that there exists an isolated singular point $(x^0, y^0) = (x_0, y_0)$, $x^0 \in [x_1, x_2]$ and $y^0 \in [y_1, y_2]$,
so that $\Delta_1(0, x^0, y^0) = 0$ and $\Delta_2(0, x^0, y^0) = 0$. This means that the system (3.1) and (4.3.2) has a periodic solutions $x(t, x_0, y_0), y(t, x_0, y_0)$ for which
$x_0 \in [a + \frac{T}{2}M_1, b - \frac{T}{2}M_1]$ and $y_0 \in [c + \frac{T}{2}M_2, d - \frac{T}{2}M_2]$. ■

Remark 3.1. Theorem 3.2 is proved when $R^n = R^1$, on the other hand as x_0, y_0 are a scalar singular point which should be isolated (For this remark, see [5]).

IV. Stability Theorem Of Solution (I_1) And (I_2)

In this section, we study theorem on stability of a periodic solution for the integral equations (I_1) and (I_2).

Theorem 4.1. If the function $\Delta_1(0, x_0, y_0), \Delta_2(0, x_0, y_0)$ are defined by equations (3.1) and (3.2), where the function $x^0(t, x_0, y_0)$ is a limit of the sequence of the functions (2.1) , the function $y^0(t, x_0, y_0)$ is the limit of the sequence of the functions (2.2) , Then the following inequalities yields:

$$\|\Delta_1(0, x_0, y_0)\| \le M_1 \qquad \cdots (4.1)$$
$$\|\Delta_2(0, x_0, y_0)\| \le M_2 \qquad \cdots (4.2)$$

and

$$\|\Delta_2(0, x_0^1, y_0{}^1) - \Delta_2(0, x_0^2, y_0{}^2)\| \le F_1F_2E_3\|F_0^1(t) - F_0^2(t)\| + \\ + \frac{T}{2}F_1F_2E_2((E_3 + E_4) + F_1E_4(1 - \frac{T}{2}E_1))\|G_0{}^1(t) - G_0{}^2(t)\| \qquad \cdots (4.4)$$

for all $x^0, x_0^1, x_0^2 \in D_{f_1}$, $y^0, y_0^1, y_0^2 \in D_{1f_2}$, and $\alpha(t) = 2t(1 - \frac{t}{T}) \le \frac{T}{2}$,
where $E_1 = [K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)], E_2 = [K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]$,

$E_3 = [L_1 + H_1(\frac{\delta}{\lambda_2} L_3 + hL_4)]$, $E_4 = [L_2 + H_2(\frac{\delta}{\lambda_2} L_3 + hL_4)]$,

$F_1 = [(1 - \frac{T}{2} E_1)(1 - \frac{T}{2} E_4)]^{-1}$ and $F_2 = (1 - \frac{T^2}{4} N_2 N_3 F_1)^{-1}$

Proof. From the properties of the functions $x^0(t, x_0, y_0)$ and $y^0(t, x_0, y_0)$ as in the theorem 4.2.1, the functions $\Delta_1 = \Delta_1(x_0, y_0)$, $\Delta_1 = \Delta_1(x_0, y_0)$,
$x_0 \in D_{f_1}, y_0 \in D_{1f_2}$ are continuous and bounded by M_1, M_2 in the domain (1.3).

From relation (3.1), we find:

$$\|\Delta_1(0, x_0, y_0\| \le \frac{1}{T}\int_0^T \|f_1(t, x^0(t, x_0, y_0), y^0(t, x_0, y_0), \int_{-\infty}^{t} G_1(t,s) g_1(s, x^0(s, x_0, y_0), y^0(s, x_0, y_0))ds, \int_{a(t)}^{b(t)} g_1(s, x^0(s, x_0, y_0), y^0(s, x_0, y_0))ds)\| \, dt$$

by using the Lemma 4.1.1, gives:

$\|\Delta_1(0, x_0, y_0)\| \le M_1$.

And from relation (3.2), we get:

$$\|\Delta_2(0, x_0, y_0\| \le \frac{1}{T}\int_0^T \|f_2(t, x^0(t, x_0, y_0), y^0(t, x_0, y_0), \int_{-\infty}^{t} G_2(t,s) g_2(s, x^0(s, x_0, y_0), y^0(s, x_0, y_0))ds, \int_{a(t)}^{b(t)} g_2(s, x^0(s, x_0, y_0), y^0(s, x_0, y_0))ds)\| \, dt$$

and using Lemma 4.1.1, we have:

$\|\Delta_2(0, x_0, y_0)\| \le M_2$

By using equation (3.1) and lemma 1.1, we get:

$$\|\Delta_1(0, x_0^1, {y_0}^1) - \Delta_1(0, x_0^2, {y_0}^2)\|$$
$$\le \frac{1}{T}\int_0^T [K_1\|x^0(t, x_0^1, {y_0}^1) - x^0(t, x_0^2, {y_0}^2)\| +$$
$$+K_2\|y^0(t, x_0^1, {y_0}^1) - y^0(t, x_0^2, {y_0}^2)\| +$$
$$+\frac{\gamma}{\lambda_1} K_3(R_1\|x^0(t, x_0^1, {y_0}^1) - x^0(t, x_0^2, {y_0}^2)\| +$$
$$+R_2\|y^0(t, x_0^1, {y_0}^1) - y^0(t, x_0^2, {y_0}^2)\|) +$$
$$+ hK_4(R_1\|x^0(t, x_0^1, {y_0}^1) - x^0(t, x_0^2, {y_0}^2)\| +$$
$$+R_2\|y^0(t, x_0^1, {y_0}^1) - y^0(t, x_0^2, {y_0}^2)\|)]dt$$
$$\le [K_1 + R_1(\frac{\gamma}{\lambda_1} K_3 + hK_4)] \, \|x^0(t, x_0^1, {y_0}^1) - x^0(t, x_0^2, {y_0}^2)\| +$$
$$+[K_2 + R_2(\frac{\gamma}{\lambda_1} K_3 + hK_4)] \, \|y^0(t, x_0^1, {y_0}^1) - y^0(t, x_0^2, {y_0}^2)\|$$

so,

$$\|\Delta_1(0, x_0^1, {y_0}^1) - \Delta_1(0, x_0^2, {y_0}^2)\| \le E_1\|x^0(t, x_0^1, {y_0}^1) - x^0(t, x_0^2, {y_0}^2)\| + E_2 \, \|y^0(t, x_0^1, {y_0}^1) - y^0(t, x_0^2, {y_0}^2)\| \quad \cdots (4.5)$$

And also by using equation (3.2) and lemma 1.1, gives:

$$\|\Delta_2(0, x_0^1, {y_0}^1) - \Delta_2(0, x_0^2, {y_0}^2)\|$$
$$\le \frac{1}{T}\int_0^T [L_1\|x^0(t, x_0^1, {y_0}^1) - x^0(t, x_0^2, {y_0}^2)\| +$$
$$+L_2\|y^0(t, x_0^1, {y_0}^1) - y^0(t, x_0^2, {y_0}^2)\| +$$

$$+\frac{\delta}{\lambda_2}L_3(H_1\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\|+$$
$$+H_2\|y^0(t,x_0^1,{y_0}^1)-y^0(t,x_0^2,{y_0}^2)\|)+$$
$$+hL_4(H_1\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\|+$$
$$+H_2\|y^0(t,x_0^1,{y_0}^1)-y^0(t,x_0^2,{y_0}^2)\|)]dt$$
$$\le[L_1+H_1(\frac{\delta}{\lambda_2}L_3+hL_4)]\ \|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\|+$$
$$+[L_2+H_2(\frac{\delta}{\lambda_2}L_3+hL_4)]\ \|y^0(t,x_0^1,{y_0}^1)-y^0(t,x_0^2,{y_0}^2)\|.$$

Therefore,

$$\|\Delta_2(0,x_0^1,{y_0}^1)-\Delta_2(0,x_0^2,{y_0}^2)\|\le E_3\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\|+$$
$$+E_4\ \|y^0(t,x_0^1,{y_0}^1)-y^0(t,x_0^2,{y_0}^2)\| \qquad \cdots(4.6)$$

where the functions $x^0(t,x_0^1,{y_0}^1)$, $y^0(t,x_0^1,{y_0}^1)$, $x^0(t,x_0^2,{y_0}^2)$ and $y^0(t,x_0^2,{y_0}^2)$ are solutions of the equation:

$$x(t,x_0^k,y_0^k)=F_0^k(t)+\int_0^t[f_1(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k),$$
$$,\int_{-\infty}^{s}G_1(s,\tau)g_1(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))\ d\tau,$$
$$,\int_{a(s)}^{b(s)}g_1(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))d\tau)-$$
$$-\frac{1}{T}\int_0^T f_1(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k),$$
$$,\int_{-\infty}^{s}G_1(s,\tau)g_1(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))\ d\tau,$$
$$,\int_{a(s)}^{b(s)}g_1(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))d\tau)ds]ds \qquad \cdots(4.7)$$

and

$$y(t,x_0^k,y_0^k)=G_0^k(t)+\int_0^t[f_2(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k),$$
$$,\int_{-\infty}^{s}G_2(s,\tau)g_2(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))\ d\tau,$$
$$,\int_{a(s)}^{b(s)}g_2(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))d\tau)-$$
$$-\frac{1}{T}\int_0^T f_2(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k),$$
$$,\int_{-\infty}^{s}G_2(s,\tau)g_2(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))\ d\tau,$$
$$,\int_{a(s)}^{b(s)}g_2(\tau,x(\tau,x_0^k,y_0^k),y(\tau,x_0^k,y_0^k))d\tau)ds]ds \qquad \cdots(4.8)$$

where $k=1,2$.

From (4.4.7), we get:

$$\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\|$$
$$\le\|F_0^1(t)-F_0^2(t)\|+$$

$$+\alpha(t)[K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\ \|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| +$$
$$+\alpha(t)[K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\ \|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\|$$
$$\le \|F_0^1(t) - F_0^2(t)\| +$$
$$+\frac{T}{2}[K_1 + R_1(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\ \|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| +$$
$$+\frac{T}{2}[K_2 + R_2(\frac{\gamma}{\lambda_1}K_3 + hK_4)]\ \|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\|$$

such that:

$$\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| \le \|F_0^1(t) - F_0^2(t)\| + \frac{T}{2}E_1\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| + \frac{T}{2}E_2\ \|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\|$$

therefore:

$$\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| \le (1-\frac{T}{2}E_1)^{-1}\|F_0^1(t) - F_0^2(t)\| + \frac{T}{2}E_2(1-\frac{T}{2}E_1)^{-1}\|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\| \quad \cdots (4.9)$$

And also from relation (4.8), we have:

$$\|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\| \le$$
$$\le \|G_0^1(t) - G_0^2(t)\| +$$
$$+\alpha(t)[L_1 + H_1(\frac{\delta}{\lambda_2}L_3 + hL_4)]\ \|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| +$$
$$+\alpha(t)[L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)]\ \|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\| \le$$
$$\le \|G_0^1(t) - G_0^2(t)\| +$$
$$+\frac{T}{2}[L_1 + H_1(\frac{\delta}{\lambda_2}L_3 + hL_4)]\ \|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| +$$
$$+\frac{T}{2}[L_2 + H_2(\frac{\delta}{\lambda_2}L_3 + hL_4)]\ \|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\|$$

so that:

$$\|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\| \le \|G_0^1(t) - G_0^2(t)\| + \frac{T}{2}E_3\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| + \frac{T}{2}E_4\ \|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\|$$

and hence:

$$\|y^0(t,x_0^1,{y_0}^1) - y^0(t,x_0^2,{y_0}^2)\| \le (1-\frac{T}{2}E_4)^{-1}\|G_0^1(t) - G_0^2(t)\| + \frac{T}{2}E_3(1-\frac{T}{2}E_4)^{-1}\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| \quad \cdots (4.10)$$

Now, by substituting inequality (4.10) in (4.9), we get:

$$\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\| \le (1-\frac{T}{2}E_1)^{-1}\|F_0^1(t) - F_0^2(t)\| +$$
$$+\frac{T}{2}E_2(1-\frac{T}{2}E_1)^{-1}[(1-\frac{T}{2}E_4)^{-1}\|G_0^1(t) - G_0^2(t)\| +$$
$$+\frac{T}{2}E_3(1-\frac{T}{2}E_4)^{-1}\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\|]$$
$$\le (1-\frac{T}{2}E_1)^{-1}\|F_0^1(t) - F_0^2(t)\| +$$
$$+\frac{T}{2}E_2[(1-\frac{T}{2}E_1)(1-\frac{T}{2}E_4)]^{-1}\|G_0^1(t) - G_0^2(t)\| +$$
$$+\frac{T^2}{4}E_2E_3[(1-\frac{T}{2}E_1)(1-\frac{T}{2}E_4)]^{-1}\|x^0(t,x_0^1,{y_0}^1) - x^0(t,x_0^2,{y_0}^2)\|]$$

putting $F_1 = [(1-\frac{T}{2}E_1)(1-\frac{T}{2}E_4)]^{-1}$,

and substituting in the last inequality, we obtain:

$$\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\| \leq (1-\frac{T}{2}E_1)^{-1}\|F_0^1(t)-F_0^2(t)\| +$$
$$+\frac{T}{2}E_2F_1\|G_0^1(t)-G_0^2(t)\| + +\frac{T^2}{4}E_2E_3F_1\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\|]$$

as the $F_1(1-\frac{T}{2}E_4)=(1-\frac{T}{2}E_1)^{-1}$

$$\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\| \leq F_1(1-\frac{T}{2}E_4)\|F_0^1(t)-F_0^2(t)\| +$$
$$+\frac{T}{2}E_2F_1\|G_0^1(t)-G_0^2(t)\| +\frac{T^2}{4}E_2E_3F_1\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\|]$$

which implies that:

$$\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\| \leq\leq F_1(1-\frac{T}{2}E_4)(1-\frac{T^2}{4}E_2E_3F_1)^{-1}\|F_0^1(t)-F_0^2(t)\| +$$
$$+\frac{T}{2}E_2F_1(1-\frac{T^2}{4}E_2E_3F_1)^{-1}\|G_0^1(t)-G_0^2(t)\|$$

putting $F_2=(1-\frac{T^2}{4}E_2E_3F_1)^{-1}$

and substituting in the last inequality, we obtain:

$$\|x^0(t,x_0^1,{y_0}^1)-x^0(t,x_0^2,{y_0}^2)\| \leq F_1F_2(1-\frac{T}{2}E_4)\|F_0^1(t)-F_0^2(t)\| +$$
$$+\frac{T}{2}E_2F_1F_2\|G_0^1(t)-G_0^2(t)\| \quad \cdots (4.11)$$

Also, substituting the inequalities (4.11) in (4.10), we find that:

$$\|y^0(t,x_0^1,{y_0}^1)-y^0(t,x_0^2,{y_0}^2)\| \leq (1-\frac{T}{2}E_4)^{-1}\|G_0^1(t)-G_0^2(t)\| +$$
$$+\frac{T}{2}E_3(1-\frac{T}{2}E_4)^{-1}[F_1F_2(1-\frac{T}{2}E_4)\|F_0^1(t)-F_0^2(t)\| +$$
$$+\frac{T}{2}E_2F_1F_2\|G_0^1(t)-G_0^2(t)\|]$$

and hence

$$\|y^0(t,x_0^1,{y_0}^1)-y^0(t,x_0^2,{y_0}^2)\| \leq \frac{T}{2}E_3F_1F_2\|F_0^1(t)-F_0^2(t)\| +$$
$$+[F_1(1-\frac{T}{2}E_1)+\frac{T}{2}F_1F_2E_2]\|G_0^1(t)-G_0^2(t)\| \quad \cdots (4.12)$$

so, substituting inequalities (4.11) and (4.12) in inequality (4.5), we get the inequality (4.3).

and the same, substituting inequalities (4.11) and (4.12) in inequality (4.6), gives the inequality (4.4). ■

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