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Upgrading Brownian motion theorems from classical stopping time towards set indexed stopping line

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Abstract: The purpose of this article is to extend numerous results from a classical stopping time in Brownian motion to a stopping line in set-indexed Brownian motion. In particular, we have advanced the following issues: hitting time, zero crossing, arcsine law, reflection principle, exiting from an interval, Wald's lemma for Brownian motion, Markov property and Doob's Optional Stopping Theorem.

Keywords: *Stopping line, stopping set, set indexed Brownian motion, flow, increasing path.*

I. INTRODUCTION

In this study, we extend the results from classical stopping time in Brownian motion to a stopping line in set indexed Brownian motion. Set indexed processes are a natural generalization of planar processes where is a collection of compact subsets of a fixed topological space (T,τ). The frame of a set-indexed Brownian motion is not only a new step towards generalization of a classical Brownian motion, but it has proved entirely new view of Brownian motion. In recent years, there have been many new results related to the dynamical properties of random processes indexed by a class of sets. A random stopping time is a central component to the study of stochastic processes. This notation was introduced by Merzbach in the plane (see [Me]), and by Saada and Slonowsky (see [Sa]) in the set indexed. Here we continue this development by proposing a notion of stopping line and stopping set in the set-indexed. Stopping time is one of the most important topics in the Brownian motion and therefore we extend this concept to stopping line in the set indexed Brownian motion for the following issues: hitting time, zero crossing, arcsine law, reflection principle, exiting from an interval, Wald's lemma for Brownian motion, Markov property and Doob's Optional Stopping Theorem.

II. PRELIMINARIES

The set-indexed framework:

Let (T,τ) denote a non-void σ-compact connected topological space. In set indexed works (see [IvMe], [Sa]), processes and filtrations will be indexed by a nonempty class $\mathbf{A}$ of compact connected subsets of T which is called an indexed collection if it satisfies the following:

1. $\varnothing \in \mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ such that $T=\bigcup_{n=1}^{\infty} B_n^{\circ}$.

2. $\mathbf{A}$ is closed under arbitrary intersections and if $A,B\in\mathbf{A}$ are nonempty, then $A\bigcap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$ and if there exists n such that $A_i\subseteq B_n$ for every i, then $\overline{\bigcup_i A_i}\in\mathbf{A}$.

3. $\sigma(\mathbf{A})=\mathbf{B}$ where $\mathbf{B}$ is the collection of Borel sets of T.

4. There existsan increasing sequence of finite sub-classes $\mathbf{A}_n=\{A_1^n,\ldots,A_{k_n}^n\}\subseteq\mathbf{A}$ closed under intersection with $\varnothing, B_n\in\mathbf{A}_n(\mathbf{u})$ ($\mathbf{A}_n(\mathbf{u})$ is the class of union of sets in $\mathbf{A}_n$), and a sequence of functions $g_n:\mathbf{A}\to\mathbf{A}_n(\mathbf{u})\cup T$ such that:

(i) g_n preserves arbitrary intersections and finite unions.

(ii) For each $A\in\mathbf{A}$, $A\subseteq g_n(A)^{\circ}$ and $A=\cap_n g_n(A)$, $g_n(A)\subseteq g_m(A)$ if $n\ge m$.

(iii) $g_n(A)\cap A'\in\mathbf{A}$ if $A,A'\in\mathbf{A}$ and $g_n(A)\cap A'\in\mathbf{A}_n$ if $A\in\mathbf{A}$ and $A'\in\mathbf{A}_n$.

(iv) $g_n(\varnothing)=\varnothing$ for all n.

(Note: $\overline{(\cdot)}$ and $(\cdot)^{\circ}$ denote respectively the closure and the interior of a set).

The choice of the collection $\mathbf{A}$ is critical: it must be sufficiently rich in order to generate the Borel sets of T , but small enough to ensure the existence of a continuous Gaussian process defined on $\mathbf{A}$. A space T cannot be discrete, and $\mathbf{A}$ is at least a continuum. Note that any collection of sets closed under intersections is a semilattice with respect to the partial order of the inclusion. For example:

a. The classical example is $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0,x] : x \in \Re_+^d\}$. This example can be extended to $T = \Re^d$ and $\mathbf{A} = \mathbf{A}(\Re^d) = \{[0,x] : x \in \Re^d\}$, which will give rise to a sort of 2^d -sides process.

b. The example (a) may be generalized as follows. Let $T = \Re_+^d$ or $T = \Re^d$ and given $\mathbf{A}$ to be the class of compact *lower sets*, i.e. the class of compact subsets A of T satisfying $t \in A$ implies $[0,t] \in \mathbf{A}$.

c. Additional examples have been given when T is a "continuous" rooted tree (see [Sl]) and T is a subspace of the Skorokhod space, $D[0,1]$ (see [IvMe]).

We will require other classes of sets generated by $\mathbf{A}$. The first is $\mathbf{A(u)}$, which is the class of finite unions of sets in $\mathbf{A}$. We note that $\mathbf{A(u)}$ is itself a lattice with the partial order induced by set inclusion. Let $\mathbf{C}$ consists of all the subsets of T of the form

$$C = A \setminus B, A \in \mathbf{A}, B \in \mathbf{A(u)}.$$

Let (Ω, F, P) denote a complete probability space. Following the established set indexed framework in [IvMe]), a set indexed filtration is any family $\{F_A : A \in \mathbf{A}\}$ of complete sub-σ-algebras of F which is increasing in the sense that $F_A \subseteq F_B$ for any $A, B \in \mathbf{A}$ such that $A \subseteq B$, and is right-continuous in the sense that $F_{\bigcap A_i} = \bigcap F_{A_i}$, for any decreasing sequence (A_i) in $\mathbf{A}$. (For consistency in what follows, if $T \notin \mathbf{A}$ define $F_T = F$).Any such filtration can be extended to $\mathbf{A(u)}$ -indexed family by definition: $F_B^\circ = \bigvee_{A \in \mathbf{A}, A \subseteq B} F_A$.

If $C \in \mathbf{C(u)} \setminus \mathbf{A}$ ($\mathbf{C(u)}$ - class of finite unions of sets in $\mathbf{C}$) then denote: $\mathbf{G}_C^* = \bigvee_{A \in \mathbf{A(u)}, A \cap C} F_A$.

In addition, let A^{ss} be any finite sub-semilattice of $\mathbf{A}$ closed under intersection. For $A \in A^{ss}$, define the left neighborhood of A in A^{ss} to be a set $C_A = A \setminus \bigcup_{B \in A^{ss}, B \subset A} B$. We note that $\bigcup_{A \in A^{ss}} A = \bigcup_{A \in A^{ss}} C_A$ and that the latter union is disjoint. The sets in A^{ss} can always be numbered in the following way: $A_0 = \varnothing'$, ($\varnothing' = \bigcap_{A \in \mathbf{A}, A \neq \varnothing} A$, note that $\varnothing' \neq \varnothing$) and given $A_0, \ldots, A_{i-1}$, choose A_i to be any set in A^{ss} such that $A \subset A_i$ implies that $A = A_j$, some $j = 1, \ldots, i-1$. Any such numbering $A^{ss} = \{A_0, \ldots, A_k\}$ will be called "consistent with the strong past" (i.e., if C_i is the left-neighborhood of A_i in A^{ss}, then $C_i = \bigcup_{j=0}^{i} A_j \setminus \bigcup_{j=0}^{i-1} A_j$ and $C_i \cap A_j = \varnothing$, for all $j = 0, \ldots, i-1,\ i = 1, \ldots, k$).

Any $\mathbf{A}$ -indexed function which has a (finitely) additive extension to $\mathbf{C}$ will be called additive (and is easily seen to be additive on $\mathbf{C(u)}$ as well). For stochastic processes, we do not necessarily require that each sample path be additive, but additivity will be imposed in an almost suresense:

A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if ithas an (almost sure) additive extension to $\mathbf{C}$: $X_\varnothing = 0$ and if $C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \bigcup C_2$ and $C_1 \bigcap C_2 = \varnothing$ then almost surely $X_C = X_{C_1} + X_{C_2}$. In particular, if $C \in \mathbf{C}$ and $C = A \setminus \bigcup_{i=1}^{n} A_i$, $A, A_1, \ldots, A_n \in \mathbf{A}$ then almost surely

$$X_C = X_A - \sum_{i=1}^{n} X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - \ldots + (-1)^n X_{A \cap \bigcap_{i=1}^{n} A_i}.$$

We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to $\mathbf{C}$ also has an (almost sure) additive extension to $\mathbf{C(u)}$.

Let $X = \{X_A : A \in \mathbf{A}\}$ be an integrable additive set-indexed stochastic process and adapted with respect to filtration $F = \{F_A : A \in \mathbf{A}\}$. X is said to be a martingale if for any $A, B \in \mathbf{A}$ such that $A \subseteq B$, we have $E[X_B \mid F_A] = X_A$.

For a study of the different kinds of martingales see [MeNu], [Za], [Kh].

III. STOPPING LINES IN THE SET-INDEXED FRAMEWORK:

In this section, we define stopping lines in the set-indexed framework. This notation was introduced first by Merzbach (see [Me]) in the plane, and by Saada and Slonowsky (see [Sa]) in the set indexed.
We use the definitions and notations from [Sa], [Me] and all this section is derivedfrom there.

We induce a natural topology on $\mathbf{A}$ via the Hausdorff metric d, defined by $d(A,B)=\inf\{\varepsilon>0: A\subseteq B^{\varepsilon}\wedge B\subseteq A^{\varepsilon}\}$, $A,B\in\mathbf{A}$, where $A^{\varepsilon}=\{x\in T: d(A,x)\le\varepsilon\}$ for any $0<\varepsilon$. If $A,B\subseteq T$ are compact, it is straightforward to show that $B\subseteq A^{\circ}$ implies $B^{\varepsilon}\subseteq A^{\circ}$ for some $0<\varepsilon$.

Definition 1. A d-closed subset L of $\mathbf{A}$ is a decreasing line if

a) Given $A,B\in L$, if $A\prec B$ or $B\prec A$, then $A=B$ (we write $A\prec B$ if $A\subseteq B^{\circ}$).

b) Given any domain $A\in\mathbf{A}$ ($A=\overline{A^{\circ}}$), if $A\notin L$, then either $A\prec L$ or $L\prec A$ (we write $A\prec L$ ($L\prec A$) if there is a set $A'\in L$ such that $A\prec A'$ (respectively, $L\prec A$)).

Let $\mathbf{L(A)}$ denote the collection of all decreasing lines in $\mathbf{A}$. Included in $\mathbf{L(A)}$ is the decreasing line at infinity, denoted L_{∞} and characterized by the property, $A\prec L_{\infty}$ for all $A\in\mathbf{A}$. Given $L\in\mathbf{L(A)}$ and $A\in\mathbf{A}$, we write $A\le L$ ($L\le A$) if $A\subseteq B$ ($B\subseteq A$) and $A<L$ ($L<A$) if $A\subset B$ ($B\subset A$) for some set $B\in L$. We write $C<L$ ($C\le L$) if there exist $A\in\mathbf{A}$ such that $C\subseteq A$, $A<L$ ($A\le L$), for some set $C\in\mathbf{C}$.

Definition 2. A map $L:\Omega\to\mathbf{L(A)}$ is set indexed stopping line (or shortly, $\mathbf{A}$-stopping line) if $[A\le L]\in F_A$ for all $A\in\mathbf{A}$. The collection of all $\mathbf{A}$-stopping lines is denoted $\mathbf{SL}$. Equivalently, if we define the random collection, $R_L=\{(A,\omega)\in\mathbf{A}\times\Omega: A\le L(\omega)\}$, then L is an $\mathbf{A}$-stopping line if $[A\in R_L]=\{\omega\in\Omega:(A,\omega)\in R_L\}\in F_A$ for every $A\in\mathbf{A}$.

Stopping line in the set-indexed Brownian motion

Definition 3. Let $S\subseteq\Re$. A (strict) increasing function $f:S\to\mathbf{A(u)}$ is called a (strict) flow. If S is an interval $[a,b]$, then f is a continuous flow if $f(s)=\overline{\bigcup_{u<s}f(u)}=\bigcap_{v>s}f(v)$ for all $s\in(a,b)$ and $f(a)=\bigcap_{v>a}f(v)$, $f(b)=\overline{\bigcup_{u<b}f(u)}$.

Note: $f:S\times\Omega\to\mathbf{A(u)}$ is a called (strict) continuous random flow if f is almost surely a (strict) increasing and continuous function.

The notion of flow was introduced in [CaWa] and used by several authors [Da], [He].

Given a set indexed stochastic process X and the flow $f:[a,b]\to\mathbf{A(u)}$, we define a process Y indexed by $[a,b]$ as follows: $Y_s=X_{f(s)}=X_s^f$ for all $s\in[a,b]$

Definition 4. A positive measure σ on $(T,\mathbf{B})$ is called strictly monotone on $\mathbf{A}$ if: $\sigma_{\varnothing'}=0$ and $\sigma_A<\sigma_B$ for all $A\subset B$, $A,B\in\mathbf{A}$. The collection of these measures is denoted by $M(\mathbf{A})$.

Definition 5. Let $\sigma\in M(\mathbf{A})$. We say that the $\mathbf{A}$-indexed process X is a Brownian motion with variance σ if X can be extended to a finitely additive process on $\mathbf{C(u)}$ and if for disjoint sets $C_1,\ldots,C_n\in\mathbf{C}$, $X_{C_1},\ldots,X_{C_n}$ are independent mean-zero Gaussian random variables with variances $\sigma_{C_1},\ldots,\sigma_{C_n}$, respectively. (For any $\sigma\in M(\mathbf{A})$, there exists a set-indexed Brownian motion with variance σ [IvMe]).

Lemma 1: Let $A^{SS}=\{\varnothing'=A_0,\ldots,A_k\}$ be any finite sub-semilattice of $\mathbf{A}$ equipped with a numbering consistent with the strong past.

Then there exists a continuous (strict) flow $f:[0,k]\to\mathbf{A}(\mathbf{u})$ such that the following are satisfied:

1. $f(0)=\emptyset'$, $f(k)=\bigcup_{j=0}^{k}A_j$
2. Each left-neighborhood C generated by A^{SS} is of the form $C=f(i)\setminus f(i-1)$ for all $1\le i\le k$.
3. If $C=f(t)\setminus f(s)$ then $C\in\mathbf{C}(\mathbf{u})$ and $F_{f(s)}\in\mathbf{G}_C^{*}$.

The proof appears in [IvMe].

Theorem 1 (The characterization of set-indexed Brownian motion by flows): Let $X=\{X_A:A\in\mathbf{A}\}$ be a square-integrable set-indexed stochastic process. Let $\sigma\in M(\mathbf{A})$ then

X is set-indexed Brownian motion with variance σ if and only if the process X^f is time-change Brownian motion for all strict continuous flows $f:[a,b]\to\mathbf{A}(\mathbf{u})$.
The proof appears in [MeYo].

From Theorem 1 and Lemma 1, we derive:
Lemma 2: Let $\sigma\in M(\mathbf{A})$ and $X=\{X_A:A\in\mathbf{A}\}$ be a square integrable set indexed stochastic process.

1. If $\{A_i\}_{i=1}^{k}$ be an increasing sequence in $\mathbf{A}(\mathbf{u})$ then there exists a strict continuous flow $f:[0,k]\to\mathbf{A}(\mathbf{u})$, $f(0)=\emptyset'$ and $f(i)=A_i$ for all $1\le i\le k$, such that X^f is a time-change Brownian motion. (The process X^f is called a time-change Brownian motion if there exists $\theta:[a,b]\to[a,b]$ such that $X^{f\circ\theta}$ is a Brownian motion, for some a strict continuous flow $f:[a,b]\to\mathbf{A}(\mathbf{u})$).
2. If $\{A_i\}_{i=1}^{\infty}$ be an increasing sequence in $\mathbf{A}(\mathbf{u})$ then there exists a strict continuous flow $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$, $f(0)=\emptyset'$ and $f(i)=A_i$ for all $1\le i$, such that X^f is a time-change Brownian motion.

The proof appears in [Yo].

Let $X=\{X_A:A\in\mathbf{A}\}$ be a set-indexed Brownian motion with variance σ. For $a>0$ define L_a to be a decreasing line in $\mathbf{A}$ such that:

a) If $A<L_a$ then $X_A<a$.

b) If $A\in L_a$ then $X_A=a$ in the first time on A.

(In other words, L_a is a collection of sets A when X reaches the value a for the first time).

Theorem 2: $L_a\in\mathbf{SL}$ (L_a is a set indexed stopping line).
Proof.
Let $A\in\mathbf{A}$. Clearly, if we know X_B for all $B\subseteq A$ then we know whether the set indexed Brownian motion had the value a before or at A ,or not. Thus, we know that $[L_a\le A]$ has occurred or not just by observing the past of the process prior to A. In other words, $[L_a\le A]=[\sup_{B\subseteq A}X_B\ge a]\in F_A$. □

Theorem 3: (Bounded stopping line) $L_a<L_\infty$ (In other words, if $A\in L_a$ then $A<L_\infty$).
Proof.
Sufficient to prove that X is almost surely unbounded, since if X hits some level b ($b\ge a$) almost surely, then by continuity and since $X_\emptyset=0$, it hits level a almost surely. I have proved in [Yo] that $\overline{\lim_{A\uparrow T}}X_A=\infty$ almost surely (for more details see [Yo]). Hence, we obtain that $L_a<L_\infty$. □

Theorem 4: (Hitting time) Let $\sigma\in M(\mathbf{A})$ and $X=\{X_A:A\in\mathbf{A}\}$ be a set-indexed Brownian motion with variance σ then

$$P[L_a \le A] = 2 - 2\Phi(\tfrac{a}{\sqrt{\sigma_A}}) = \tfrac{2}{\sqrt{2\pi\sigma_A}} \int_a^{\infty} e^{-\frac{x^2}{2\sigma_A}} dx \text{ for all } A \in \mathbf{A}$$

(Φ - standard Gaussian distribution function).

Proof.

Let $A \in \mathbf{A}$. Based on a Law of Total Probability:

$$P[X_A \ge a] = P[X_A \ge a \mid A < L_a] P[A < L_a] + P[X_A \ge a \mid L_a \le A] P[L_a \le A]$$

According to definition of L_a, we imply that if $A < L_a$ then $X_A < a$, and then $P[X_A \ge a \mid A < L_a] = 0$. According to Lemma 2, there exists a strict continuous flow $f:[0,\infty) \to \mathbf{A(u)}$ and there exists $0 < t$ such that $f(t) = A$ and X^f is a time-change Brownian motion (In other words, there exists increasing function $\theta:[0,\infty) \to [0,\infty)$ such that $X^{f\circ\theta}$ is a Brownian motion). Since, $X^{f\circ\theta}$ is symmetric then clearly:

$$P[X_A \ge a \mid L_a \le A] = P[X_t^{f\circ\theta} \ge a \mid L_a \le A] = \tfrac{1}{2}$$

Then

$$P[L_a \le A] = 2P[X_A \ge a] = 2 - 2\Phi(\tfrac{a}{\sqrt{\sigma_A}}) = \tfrac{2}{\sqrt{2\pi\sigma_A}} \int_a^{\infty} e^{-\frac{x^2}{2\sigma_A}} dx. \ \square$$

Theorem 5: (Zero crossing) Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion with variance σ. Let $A \in \mathbf{A}$, for any $b \ne 0$ ($b \in \Re$), the probability that $X^b = \{X_A^b : A \in \mathbf{A}\} = \{X_A + b : A \in \mathbf{A}\}$ has at least one zero on the set $\{B \in \mathbf{A} : \varnothing \ne B \subseteq A\}$ is the same probability of $P[L_{|b|} \le A]$ (In other words, $P[X^b$ has at least a zero between $\varnothing$ and $A \in \mathbf{A}]$ = $P[L_{|b|} \le A]$).

Proof.

If $b < 0$, the due to continuity of X^b, the events $\{X^b$ has at least a zero between $\varnothing$ and $A \in \mathbf{A}\}$ and $\{\sup_{\varnothing \ne B \subseteq A} X_B^b \ge 0\}$ are the same then $P[X^b$ has at least a zero between $\varnothing$ and $A \in \mathbf{A}] = P[\sup_{\varnothing \ne B \subseteq A} X_B^b \ge 0] =$ $P[\sup_{\varnothing \ne B \subseteq A} X_B + b \ge 0] = P[\sup_{\varnothing \ne B \subseteq A} X_B \ge -b]$.

According to Theorems 2 and 4, we imply that $P[\sup_{\varnothing \ne B \subseteq A} X_B \ge -b] = 2P[X_B \ge -b] = P[L_{-b} \le A]$.

If $b > 0$ then $-X_A^b = -X_A - b$ is a set indexed Brownian motion and by symmetry of Brownian motion we obtain that $P[X^b$ has at least a zero between $\varnothing$ and $A \in \mathbf{A}] = P[L_{|b|} \le A]$. $\square$

Theorem 6: (Arcsine law for Brownian motion) Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion with variance σ. Denote the event $O(B,A) = [X$ has at least one zero on the set $A \setminus B]$, $A \subseteq B$, $A, B \in \mathbf{A}$ then $P[O(B,A)] = 1 - \frac{2}{\pi}\arcsin\left(\sqrt{\frac{\sigma_B}{\sigma_A}}\right)$.

Proof.

Based on Law of Total Probability, $P[O(B,A)] = \frac{1}{\sqrt{2\pi\sigma_B}} \int_{-\infty}^{\infty} P[O(B,A) \mid X_B = a] e^{-\frac{a^2}{2\sigma_B}} da$. By symmetry of Brownian motion and by $[L_a \le A] = [\sup_{B \subseteq A} X_B \ge a]$ we obtain that: $P[O(B,A) \mid X_B = a] = [L_{|a|} \le A]$.

According to Theorem 4, $P[L_{|a|} \le A] = 2 - 2\Phi(\frac{|a|}{\sqrt{\sigma_A}})$, then

$$P[O(B,A)] = \tfrac{1}{\sqrt{2\pi\sigma_B}} \int_{-\infty}^{\infty} \tfrac{2}{\sqrt{2\pi\sigma_A}} \int_{|a|}^{\infty} e^{-\frac{x^2}{2\sigma_A}} dx e^{-\frac{a^2}{2\sigma_B}} da = 1 - \tfrac{2}{\pi}\arcsin\left(\sqrt{\tfrac{\sigma_B}{\sigma_A}}\right). \ \square$$

Theorem 7: (Reflection principle) Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion with variance σ then $W_A = \begin{cases} X_A & ,A < L_a \\ 2a - X_A & ,A \geq L_a \end{cases}$ is a set indexed Brownian motion with variance σ.

Proof.

It must be shown that if $\{C_1, C_2, ..., C_k\} \in \mathbf{C}$ are disjoint, and then $W_{C_1}, W_{C_2}, ..., W_{C_k}$ are independent normal random variables with variances $\sigma_{C_1}, \sigma_{C_2}, ..., \sigma_{C_k}$, respectively. Let $A \in L_a$, without loss of generality, we may assume that the sets $\{C_1, C_2, ..., C_k\}$ are the left neighborhoods of the sub semi-lattice A^{ss} of $\mathbf{A}$ equipped with a numbering consistent with the strong past. According to Lemma 1 and Lemma 2, there exists a strict continuous flow $f : [0, \infty) \to \mathbf{A}(\mathbf{u})$ and $t_a \geq 0$ such that X^f is a time-change Brownian motion (In other words, there exists increasing function $\theta : [0, \infty) \to [0, \infty)$ such that $X^{f \circ \theta}$ is a Brownian motion) and each left-neighborhood generated by A^{ss} is of the form $C_i = f(i) \setminus f(i-1), i = 1, 2, ..., k$ and $f(\theta(t_a)) = A \in L_a$. We recall that, if $X = \{X_t : t \geq 0\}$ is a classical Brownian motion and $T_a = \inf\{t \geq 0 : X_t = a\}$, then $Z_t = \begin{cases} X_t & ,t < T_a \\ 2a - X_t & ,t \geq T_a \end{cases}$ is a Brownian motion [Fr]. Thus, if we define

$$W_t = \begin{cases} X_t^{f \circ \theta} & ,t < t_a \\ 2a - X_t^{f \circ \theta} & ,t \geq t_a \end{cases},$$

W_t turns out to be a Brownian motion, and so $W_{C_1}, W_{C_2}, ..., W_{C_k}$ are independent normal random variables with variances $\sigma_{C_1}, \sigma_{C_2}, ..., \sigma_{C_k}$, respectively.□

Theorem 8: (Exiting from an interval) Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion with variance σ. Define the exit set by $T_D = \bigcap_{A \in D(a,b)} A$ then

a. T_D is a stopping set (A random set $\xi : \Omega \to \mathbf{A}(\mathbf{u})$ is called a stopping set if $[A \subseteq \xi] \in F_A$ for any $A \in \mathbf{A}$. For further details see [Sa], [IvMe95])

b. $P[X_{T_D} = b] = \frac{|a|}{b+|a|}$

When $D(a,b) = [A \in \mathbf{A} : X_A \notin (a,b)$ for the first time $] = \{(A, \omega) \in \mathbf{A} \times \Omega : X_A(\omega) \notin (a,b)$ for the first time$\}$ for $a < 0 < b$. (In other words, $D(a,b)$ to be a collection of sets $A \in \mathbf{A}$ such that if $A \in D(a,b)$ then $X_A \notin (a,b)$ for the first time on A).

Proof.

a. Let $A \in \mathbf{A}$,

$$[T_D \subseteq A]^c = [T_D \supset A] = [X_B \in (a,b) \text{ for all } B \subseteq A] \in F_A$$

Then T_D is a stopping set.

b. According to Lemma 1 and Lemma 2, there exists a strict continuous random flow $f : [0, \infty) \times \Omega \to \mathbf{A}(\mathbf{u})$ and $t_D \geq 0$ (t_D is a stopping time) such that X^f is a time-change Brownian motion and $T_D = f(t_D)$. (In other words, there exists increasing function $\theta : [0, \infty) \times \Omega \to [0, \infty) \times \Omega$ and $t_\theta \geq 0$ (t_θ is a stopping time) such that $X^{f \circ \theta}$ is a Brownian motion and $T_D = f(t_D) = f(\theta(t_\theta))$). We recall that, if $X = \{X_t : t \geq 0\}$ is a classical Brownian motion and $T(a,b) = \inf\{t \geq 0 : X_t \notin (a,b)\}$ ($a < 0 < b$), then $P[X_{T(a,b)} = b] = \frac{|a|}{b+|a|}$.

Thus, $P[X_{T_D} = b] = P[X^f_{t_D} = b] = P[X^{f \circ \theta}_{t_\theta} = b] = \frac{|a|}{b+|a|}$. □

Theorem 9: (Markov property) Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed Brownian motion with variance σ.

a. If $B \in \mathbf{A}$ then $W = \{W_A : A \in \mathbf{A}\}$ is a set-indexed Brownian motion with variance σ when $W_A = X_{B\cup A} - X_{B\setminus A}$.

b. If $g \in G^{\uparrow}$ then $W = \{W_A : A \in \mathbf{A}\}$ is a set-indexed Brownian motion with variance σ when $W_A = X_{g*A} - X_{g*A\setminus A}$ and $G^{\uparrow} = \{g \in G : \forall A, B \in A, A \subseteq B \Rightarrow g * A \subseteq g * B\}$

(Let $\sigma \in M(\mathbf{A})$ and $(G,\cdot)$ is a group. A group action $*$ of $(G,\cdot)$ on $\mathbf{A}$ is defined by: $g*(A\cup B) = g*A \cup g*B,\ g*(A\setminus B) = g*A \setminus g*B$ for all $A, B \in \mathbf{A}, g \in G$ and there exist $\eta : G \to \Re_{+}$ such that $\sigma(g*A) = \eta(g)\sigma(A)$ for all $A \in \mathbf{A}, g \in G$.

(The definition and more details about group action on $\mathbf{A}$ appears in [Yo15]).

Proof.

Easy to see that W can be extended to a finitely additive process on $\mathbf{C(u)}$ and for disjoint sets $C_1, \ldots, C_n \in \mathbf{C}$, $X_{C_1}, \ldots, X_{C_n}$ are independent mean-zero Gaussian random variables. Enough to prove that $Var(W_A) = \sigma_A$ for all $A \in \mathbf{A}$.

a. $Var(W_A) = Var(X_{B\cup A} - X_{B\setminus A}) = Var(X_{B\cup A}) + Var(X_{B\setminus A}) - 2E[X_{B\cup A}X_{B\setminus A}] =$
$= \sigma_{B\cup A} + \sigma_{B\setminus A} - 2\sigma_{(B\cup A)\cap(B\setminus A)} = \sigma_{B\cup A} + \sigma_{B\setminus A} - 2\sigma_{B\setminus A} = \sigma_{B\cup A} - \sigma_{B\setminus A} = \sigma_A$.

b. $Var(W_A) = Var(X_{g*A} - X_{g*A\setminus A}) = Var(X_{g*A}) + Var(X_{g*A\setminus A}) - 2E[X_{g*A}X_{g*A\setminus A}] =$
$= \sigma_{g*A} + \sigma_{g*A\setminus A} - 2\sigma_{(g*A)\cap(g*A\setminus A)} = \sigma_{g*A} + \sigma_{g*A\setminus A} - 2\sigma_{g*A\setminus A} = \sigma_{g*A} - \sigma_{g*A\setminus A} =$
$= \sigma_{g*A} - (\sigma_{g*A} - \sigma_A) = \sigma_A$. □

Theorem 10: (Wald's lemma for Brownian motion) Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed Brownian motion with variance σ and $L \in \mathbf{SL}$ then for all $\xi \in L$

a. $EX_{\xi}^2 = E[\langle X \rangle_{\xi}] = E[\sigma_{\xi}]$

b. There exists a strict continuous random flow $f : [0,\infty) \times \Omega \to \mathbf{A(u)}$ and a classical stopping time $\tau = f^{-1}[\xi]$ (or a stopping set $f(\tau) = \xi$) such that $EX_{\xi} = 0,\ EX_{\xi}^2 = E[\tau]$.

(Note: Let $X = \{X_t : t \geq 0\}$ be a square integrable martingale. It is known that we can associate with X a unique predictable process denoted $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a martingale.

Under some hypothesis, we can define $\langle X \rangle$ to be a compensator associated with the sub martingale X^2. The definition and more details about $\langle X \rangle$ can be found in [IvMe]).

Proof.

Let $\xi \in L$. Based on Lemma 2, there exists a strict continuous random flow $f : [0,\infty) \times \Omega \to \mathbf{A(u)}$ and stopping time $0 < \tau < \infty$, such that X^f is a time-change Brownian motion and $f(\tau) = \xi$.

a. X is a set-indexed Brownian motion with variance σ then $\langle X \rangle_A = \sigma_A$ and $X^2 - \langle X \rangle$ is a martingale (the proof appears in [MeYo]). $X^2 - \langle X \rangle$ is a martingale then $(X^f)^2 - \langle X \rangle^f$ is a martingale (see [IvMe], [MeYo]). Doob's Optional Stopping Theorem (We recall that, let $M = \{M_t : t \geq 0\}$ be a martingale with filtration $F = \{F_t : t \geq 0\}$. If T is a bounded stopping time then $E[M_T] = M_0$) we imply that $E[(X_{\tau}^f)^2] = E[\langle X \rangle_{\tau}^f] \Rightarrow\ EX_{\xi}^2 = E[\langle X \rangle_{\xi}] = E[\sigma_{\xi}]$.

b. X^f is a time-change Brownian motion then there exists increasing function $\theta:\Omega\times[0,\infty)\to\Omega\times[0,\infty)$ and $0<\alpha<\infty$, such that $X^{f\circ\theta}$ is a Brownian motion and $f(\tau)=f(\theta(\alpha))=\xi$. L is a set indexed stopping line then α,τ are a stopping times and $f(\tau)=f(\theta(\alpha))=\xi$ is a stopping set, so there is a nonrandom $N<\infty$ such that $\alpha<N$ almost surely. By the Strong Markov Property, the process $X^{f\circ\theta}_{t+\alpha}-X^{f\circ\theta}_{\alpha}$ is a standard Wiener process that is independent of the stopping field $F^{f\circ\theta}_{\alpha}$, and, in particular, independent of a random vector $(\theta(\alpha),X^{f\circ\theta}_{\alpha})$. Hence, the conditional distribution of $X^{f\circ\theta}_{N}-X^{f\circ\theta}_{\alpha}$ given that $EX_{\xi}=E[X^{f\circ\theta}_{\alpha}]=0$ and $E[(X^{f\circ\theta}_{\alpha})^2]=E[(X^{f}_{\theta(\alpha)})^2]=E[\theta(\alpha)]=E[\tau]\Rightarrow EX^2_{\xi}=E[\tau]$. □

Theorem 11: (Doob's Optional Stopping Theorem) Let $\sigma\in M(\mathbf{A})$ and $X=\{X_A:A\in\mathbf{A}\}$ be a set-indexed Brownian motion with variance σ. Then for bounded set indexed stopping line $L,S\in\mathbf{SL}$ with $S\le L$ almost surely we have that $E[X_{\xi}\mid F_{\eta}]=X_{\eta}$ for all $\xi\in S,\ \eta\in L$.

Proof.

Let $\xi\in S,\ \eta\in L$. According to Lemma 2, there exists a strict continuous random flow $f:[0,\infty)\times\Omega\to\mathbf{A(u)}$ and stopping times $0<\tau\le\mu<\infty$, such that X^f is a time-change Brownian motion and $f(\tau)=\xi,\ \ f(\mu)=\eta$. X^f is a time-change Brownian then there exists increasing function $\theta:\Omega\times[0,\infty)\to\Omega\times[0,\infty)$ and $0<t\le m<\infty$, such that $X^{f\circ\theta}$ is a Brownian motion and $f(\tau)=f(\theta(t))=\xi,\ \ f(\mu)=f(\theta(m))=\eta$. $X^{f\circ\theta}$ is a Brownian motion then $X^{f\circ\theta}$ is a martingale. Based on Doob's Optional Stopping Theorem (We recall that, let $M=\{M_t:t\ge0\}$ be a martingale with filtration $F=\{F_t:t\ge0\}$. If S and T are a bounded stopping times with $S\le T$ almost surely then $E[M_T\mid F_S]=M_S$ almost surely), we imply that $E[X^{f\circ\theta}_t\mid F^{f\circ\theta}_m]=X^{f\circ\theta}_m\Rightarrow E[X_{f(\theta(t))}\mid F_{f(\theta(m))}]=X_{f(\theta(m))}\Rightarrow E[X_{\xi}\mid F_{\eta}]=X_{\eta}$. □

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