

Spanning Trees of Ladders

M.Lewinter and S.Caraiani

Abstract

We find spanning trees of the ladder, $M(n, 2)$. These include the path, P_{2n} , equitable 3-branched starlike and double starlike trees, and others.

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Given positive integers $a, b \geq 2$, the mesh $M(a, b) = P_a \times P_b$. In particular, $M(n, 2)$, where $n \geq 3$, is called a ladder. [1]

A tree is a connected, acyclic graph. It is called *starlike*, if it contains a vertex v for which $\deg(v) \geq 3$, and the other vertices in T have degree ≤ 2 . The vertex v is called the *junction*. If $\deg(v) = k$, we say T is *k-branched*. $T - v$ has k components, each of which is either K_1 , K_2 , or a path. Let $S(a_1, a_2, a_3)$ denote a 3-branched starlike tree T with junction v such that the components of $T - v$ are paths of orders a_1 , a_2 , and a_3 . See Figure 1. The junction is labeled v .

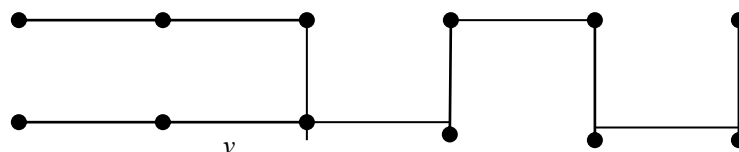


Figure 1: The 3-branched starlike tree $S(2, 3, 6)$ spans $M(6, 2)$

A *bipartite* graph has the property that each of its vertices is colored black or white, and adjacent vertices are oppositely colored. A bipartite graph is *equitable* if the two color sets have the same cardinality. In particular, ladders are equitably bipartite. We have the following theorem. The proof is left to the reader.

Theorem 1: If the tree, T , spans the ladder, $M(n, 2)$, **(a)** it must have order $2n$, **(b)** it must be equitable, and **(c)** its maximum degree cannot exceed 3. ■

A 3-branched starlike tree T is equitable if and only if exactly two of its endvertices have the same color as the junction. Equivalently, a 3-branched starlike tree T with junction v , is equitable if and only if exactly two of the paths of $T - v$ are even. We have the following theorem.

Theorem 2: An equitable, 3-branched, starlike tree on $2n$ vertices, where $n \geq 3$, spans the ladder $M(n, 2)$.

Proof: Let T be an equitable 3-branched starlike tree on $2n$ vertices. Let the junction be v and let the three endvertices be x, y , and z , respectively. Assume, WLOG, that the junction v and the endnode z are black, while the remaining vertices x and y are oppositely colored. Let R be the unicyclic graph $T \cup xy$. Embed the cycle of R as shown in Figure 2, and embed the path with endnode z in the winding manner shown there. This last embedding is achievable since v and z have the same color. Upon deleting edge xy , we find that T spans $M(n, 2)$. The same holds if edge xy is horizontal. ■

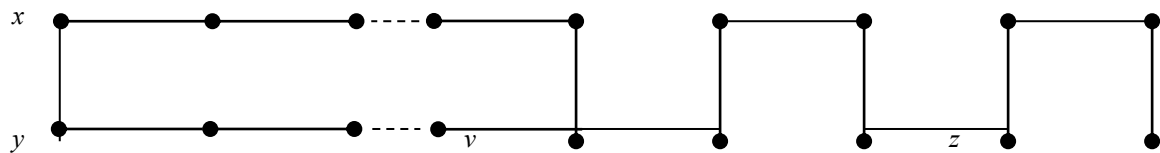


Figure 2: The unicyclic graph $T \cup xy$ spans a ladder.

A tree such that two adjacent vertices, say u and v , have degree ≥ 3 while the remaining vertices have degrees 2 or 1, is a *double starlike tree*. In particular, if u and v , have degree 3, we denote it $DS(a, b; c, d)$ where one vertex of degree three has branches of length a and b , while the other vertex of degree three has branches of length c and d . We do not consider uv a branch. Since $M(n, 2)$ is equitable, we have the following theorem whose proof is left to the reader.

Theorem 3: $DS(a, b; c, d)$ is equitable if and only if the four endnodes form an equitable set. ■ We present the final theorem whose proof is left to the reader after generalizing Figure 3.

Theorem 4: $DS(a, b; c, d)$, such that $a + b$ and $c + d$ are odd, spans the ladder $M\left(\frac{a + b + c + d + 2}{2}, 2\right)$. ■

Example:

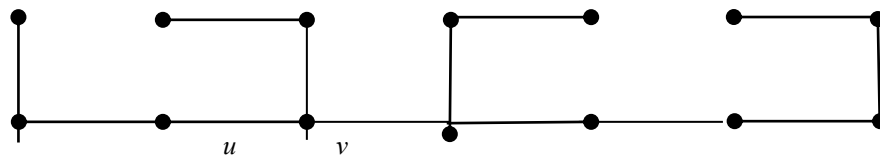


Figure 3: $DS(2, 3; 5, 2)$ spans $M(6, 2)$

References

- [1] F.Buckley and M.Lewinter, *A Friendly Introduction to Graph Theory*. Prentice-Hall, 2003.
- [2] F.Buckley, M.Lewinter, *Introductory Graph Theory with Applications*. Waveland Press. 2014.