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Comparative Performance Evaluation of Hermitian Covariance Matrix Estimators under the Circularly Symmetric Complex Normal Distribution: A Monte Carlo Simulation Study

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Abstract
Accurate estimation of Hermitian covariance matrices plays a fundamental role in multivariate signal processing, wireless communications, radar systems, and complex-valued statistical analysis. This study presents a comprehensive Monte Carlo simulation framework for evaluating the performance of four widely used covariance matrix estimators under the Circularly Symmetric Complex Normal (CSCN) distribution: the Sample Covariance Matrix (SCM), Ledoit–Wolf shrinkage estimator, Oracle Approximating Shrinkage (OAS) estimator, and Tyler's M-estimator. Hermitian positive-definite Toeplitz covariance matrices were used as the population covariance structure, while sample sizes ranging from 25 to 1000 were considered. Each simulation scenario was repeated 1000 times to ensure stable performance assessment.
The estimators were evaluated using Bias, Mean Squared Error (MSE), Relative Frobenius Error (RFE), Eigenvalue Error (EE), and computational runtime. The simulation results demonstrated that estimation accuracy improved consistently with increasing sample size for all estimators. Ledoit–Wolf and OAS shrinkage estimators achieved lower MSE and RFE than the SCM estimator, particularly for small and moderate sample sizes, whereas Tyler's M-estimator provided competitive eigenvalue estimation but required substantially longer computation due to its iterative estimation procedure. As the sample size increased, the performance differences among the estimators gradually diminished.
The findings provide practical guidance for selecting covariance matrix estimators in complex-valued statistical applications by highlighting the trade-off between estimation accuracy and computational efficiency under the CSCN model.
Keywords: *Hermitian matrices, Complex Normal distribution, simulation.*

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I. Introduction

Complex-valued random vectors constitute the mathematical foundation of numerous modern signal processing and statistical inference problems. Unlike real-valued observations, complex-valued data simultaneously represent amplitude and phase information, enabling more accurate modeling of oscillatory and wave-based phenomena. Consequently, complex-valued statistical models have become indispensable in radar systems, wireless communications, sonar, synthetic aperture radar (SAR), magnetic resonance imaging (MRI), array signal processing, adaptive beamforming, and biomedical signal analysis. In these applications, preserving the intrinsic complex structure of the observations is essential for maintaining both statistical efficiency and physical interpretability. As a result, the development of reliable statistical methods for analyzing complex-valued data has attracted considerable attention over the past several decades (Goodman, 1963; Picinbono, 1996; Schreier & Scharf, 2010).

Among the various probabilistic models for complex-valued observations, the Circularly Symmetric Complex Normal (CSCN) distribution represents the most fundamental and widely adopted framework. A complex random vector is said to be circularly symmetric when its probability distribution remains invariant under arbitrary phase rotations, implying that the statistical properties are independent of the global phase angle. This property considerably simplifies theoretical analysis because the pseudo-covariance matrix vanishes, leaving the Hermitian covariance matrix as the complete second-order statistical descriptor. Owing to these mathematical properties, the CSCN distribution has become the standard model in communication theory, array processing, statistical signal processing, and multivariate complex analysis, where Gaussian assumptions often provide accurate approximations for practical systems (Schreier & Scharf, 2010; Ollila, Tyler, Koivunen, & Poor, 2012).

The covariance matrix plays a central role in multivariate statistical analysis because it characterizes the dependence structure among variables and forms the basis of numerous estimation and detection algorithms. In the complex-valued setting, this covariance matrix must satisfy the Hermitian property, ensuring conjugate symmetry and positive definiteness. Accurate estimation of Hermitian covariance matrices is therefore a prerequisite for a broad range of applications, including principal component analysis, adaptive filtering, Kalman filtering, direction-of-arrival estimation, multiple-input multiple-output (MIMO) communication systems, interference suppression, and subspace-based signal detection algorithms. The overall performance of these techniques depends heavily on the quality of covariance estimation, making the development of efficient Hermitian covariance estimators an important research topic in both statistics and engineering (Kay, 1993; Stoica & Moses, 2005).

Despite its fundamental importance, covariance estimation remains challenging, particularly when the available sample size is limited or the dimensionality of the data increases. Under such circumstances, estimation variability increases substantially, resulting in unstable eigenvalue estimates, ill-conditioned covariance matrices, and degraded statistical performance. These limitations have motivated extensive research into alternative covariance estimation techniques that improve estimation accuracy while maintaining computational feasibility. Consequently, covariance matrix estimation continues to be an active area of research in multivariate statistics, statistical signal processing, and high-dimensional data analysis.

To compare the statistical performance of different Hermitian covariance matrix estimators under Circularly Symmetric Complex Normal (CSCN) distributions using extensive Monte Carlo simulations.

II. Material and Methods

Circularly Symmetric Complex Normal Distribution

Let $z \in \mathbb{C}^p$ be a complex random vector following the Circularly Symmetric Complex Normal (CSCN) distribution, denoted by $z \sim \mathcal{CN}_p(\mu, \Sigma)$. Under circular symmetry, the pseudo-covariance matrix satisfies $E[(z-\mu)(z-\mu)^T] = 0$, and the covariance matrix is Hermitian positive definite (Schreier and Scharf, 2010).

$$f(z) = \frac{1}{\pi^p \det(\Sigma)} \exp[-(z-\mu)^H \Sigma^{-1}(z-\mu)]$$

Hermitian Toeplitz Covariance Matrix

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ \rho^* & 1 & \rho & \cdots & \rho^{p-2} \\ \vdots & & \ddots & & \vdots \\ (\rho^*)^{p-1} & \cdots & \rho^* & 1 & \end{bmatrix}$$

Toeplitz covariance matrices are widely used in radar, array processing, and wireless communication because they represent stationary correlation structures among neighboring sensors (Kay, 1993).

Sample Covariance Matrix (SCM)

The sample Hermitian covariance matrix for the complex observations

$$Z_1, Z_2, \ldots, Z_n \sim \mathcal{CN}_p(\mu, \Sigma)$$

obtained from the Circularly Symmetric Complex Normal (CSCN) distribution is defined as:

$$\hat{\Sigma}_{SCM} = \frac{1}{n-1}\sum_{i=1}^{n}(z_i - \bar{z})(z_i - \bar{z})^H$$

Here: Hermitian (conjugate transpose), $\bar{z}$:sample mean, Σ: true covariance matrix. This estimator is the most commonly used covariance estimator in complex statistics. SCM has a complex Wishart distribution (Goodman, 1963).

The basic characteristics of SCM are as follows:

1. Hermitian structure

$$\hat{\Sigma} = \hat{\Sigma}^H$$

2. Positive definite

If $n > p$ then $\hat{\Sigma} > 0$.

3. Unbiased

Under the CSCN assumption

$$E(\hat{\Sigma}) = \Sigma$$

that is, it is unbiased.

4. Consistency

$$\hat{\Sigma} \underset{n\to\infty}{\overset{P}{\rightarrow}} \Sigma$$

Therefore, as the sample size increases, the error should decrease.

Ledoit–Wolf Shrinkage Estimator
The mathematical formula is as follows.

$$\hat{\Sigma}_{LW} = (1-\lambda)\hat{\Sigma}_{SCM} + \lambda T$$

Here, T is the target matrix (typically μI).
The shrinkage parameter $\lambda$is estimated analytically to minimize the expected Frobenius norm loss (Ledoit and Wolf, 2004).
Oracle Approximating Shrinkage (OAS)
The formula:

$$\hat{\Sigma}_{OAS} = (1-\lambda_{OAS})\hat{\Sigma}_{SCM} + \lambda_{OAS}\mu I$$

OAS provides an approximation to the oracle shrinkage coefficient under Gaussian assumptions and generally performs better than Ledoit–Wolf for very small sample sizes (Chen et al., 2010).
Tyler's M-Estimator
The formula:

$$\hat{\Sigma} = \frac{p}{n}\sum_{i=1}^{n}\frac{z_i z_i^H}{z_i^H \hat{\Sigma}^{-1} z_i}$$

Tyler's estimator is a distribution-free robust covariance estimator that is particularly useful under elliptical distributions and heavy-tailed data (Tyler, 1987; Ollila et al., 2012).
Performance Measures
Bias

$$Bias = \frac{1}{R}\sum_{r=1}^{R}(\hat{\Sigma}_r - \Sigma)$$

MSE

$$MSE = \frac{1}{R}\sum_{r=1}^{R}\| \hat{\Sigma}_r - \Sigma \|_F^2$$

Relative Frobenius Error

$$RFE = \frac{\| \hat{\Sigma} - \Sigma \|_F}{\| \Sigma \|_F}$$

Eigenvalue Error

$$EE = \frac{1}{p}\sum_{j=1}^{p} | \hat{\lambda}_j - \lambda_j |$$

Runtime

$$Runtime = t_{end} - t_{start}$$

Monte Carlo Simulation Design
For each sample size (n = 25, 50, 100, 500, and 1000), 1000 independent Monte Carlo replications were generated from the CSCN distribution. In each replication, the SCM, Ledoit–Wolf, OAS, and Tyler covariance estimators were computed. Estimation performance was evaluated by comparing the estimated covariance matrix with the true population covariance matrix. The simulation procedure was implemented in R software (Version 4.5.1) using a fixed random seed to ensure reproducibility (R Core Team, 2025).
The population covariance matrix was specified as a Hermitian positive-definite Toeplitz matrix defined by

$$\Sigma_{ij} = \rho^{|i-j|},\ i,j = 1,\ldots,p,$$

where $\rho = 0.5$denotes the correlation coefficient between adjacent variables. This covariance structure is widely used in Monte Carlo studies because it represents exponentially decaying correlations while preserving Hermitian positive definiteness.

III. Results

If the following Toeplitz matrix was defined as the population covariance matrix for the simulation, all results in Table 1–Table 5 were obtained according to this matrix.That is, the actual covariance matrix you used is:

$$\Sigma = \begin{bmatrix} 1 & 0.5 & 0.25 & 0.125 & 0.0625 \\ 0.5 & 1 & 0.5 & 0.25 & 0.125 \\ 0.25 & 0.5 & 1 & 0.5 & 0.25 \\ 0.125 & 0.25 & 0.5 & 1 & 0.5 \\ 0.0625 & 0.125 & 0.25 & 0.5 & 1 \end{bmatrix}$$

This matrix is obtained from the following formula:

$$\Sigma_{ij} = \rho^{|i-j|},\ \rho = 0.5.$$

Compound Symmetry

$$\Sigma_{ij} = \begin{cases} 1, & i = j \\ \rho, & i \neq j \end{cases}$$

$$\rho = 0.5$$

To illustrate

$$\Sigma = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}$$

Hermitian covariance matrices are constructed.All eigenvalues are positive.

$$\lambda = (2.2619\ 1.2303\ 0.6903\ 0.4572\ 0.3602)$$

Therefore,

$$\Sigma > 0$$

is confirmed.
Condition Number

$$\kappa(\Sigma) = 6.895$$

Frobenius Norm

$$\| \Sigma \|_F = 2.72861$$

This value will then form the denominator in the Relative Frobenius Error calculation. That is,

$$RFE = \frac{\| \hat{\Sigma} - \Sigma \|_F}{\| \Sigma \|_F}$$

is calculated.
Monte Carlo simulation was performed to determine the Hermitian covariance matrix estimators under the CSCN distribution, and the results are presented in Table 1.

Table 1. Monte Carlo simulation design for evaluating Hermitian covariance matrix estimators under the CSCN distribution

Component	**Description**
Distribution	Circularly Symmetric Complex Normal (CSCN)
Mean Vector	$\mu = 0$
True Covariance Matrix	Hermitian positive-definite Toeplitz covariance matrix
Covariance Structure	$\Sigma_{ij} = \rho^{|i-j|}$
Matrix Dimensions (p)	2, 5, 10, 20
Sample Sizes (n)	25, 50, 100, 500, 1000
Number of Monte Carlo Replications	1000
Covariance Estimators	Sample Covariance Matrix (SCM), Ledoit–Wolf Shrinkage, Oracle Approximating Shrinkage (OAS), Tyler's M-estimator
Performance Criteria	Bias, Mean Squared Error (MSE), Relative Frobenius Error (RFE), Eigenvalue Error (EE), Computational Runtime
Programming Environment	R (version 4.5.1)
Random Seed	123

The Monte Carlo performance of the covariance matrix (p = 5, R = 1000) was tested, and the results are presented in Table 2.

Table 2. Monte Carlo performance of the Sample Covariance Matrix (SCM) under the CSCN distribution (p = 5, R = 1000)

Sample Size (n)	Bias	Mean MSE	SD MSE	Mean RFE	SD RFE	Mean EE	SD EE	Runtime (s)
25	0.01157	1.04508	0.46011	0.36653	0.07763	0.83228	0.35054	0.436
50	0.00709	0.50707	0.22940	0.25547	0.05331	0.56399	0.23239	0.440

100	0.00419	0.25169	0.11010	0.17991	0.03792	0.39898	0.16484	0.500
500	0.00278	0.05117	0.02235	0.08113	0.01707	0.18034	0.07417	0.777
1000	0.00129	0.02451	0.01014	0.05622	0.01149	0.12415	0.05089	1.049

As shown in Table 2, the performance of the Sample Covariance Matrix (SCM) estimator improved consistently with increasing sample size. The Bias decreased from 0.01157 at $n = 25$to 0.00129 at $n = 1000$, indicating asymptotic consistency. Similarly, the Mean Squared Error (MSE), Relative Frobenius Error (RFE), and Eigenvalue Error (EE) all declined substantially as the sample size increased. Although the computational time increased moderately with larger sample sizes, SCM remained computationally efficient throughout all simulation scenarios.
The Monte Carlo results of the Ledoit-Wolf contractional covariance estimator are shown in Table 3.

Table 3. Monte Carlo performance of the Ledoit–Wolf shrinkage covariance estimator under the CSCN distribution (p = 5, R = 1000)

Sample Size (n)	Bias	Mean MSE	SD MSE	Mean RFE	SD RFE	Mean EE	SD EE	Runtime (s)
25	0.16615	0.83333	0.33168	0.32821	0.06486	0.89242	0.36492	0.617
50	0.10197	0.45736	0.20183	0.24251	0.05121	0.60378	0.25298	0.764
100	0.05439	0.23874	0.10148	0.17534	0.03640	0.41727	0.17030	1.013
500	0.01100	0.05044	0.02197	0.08055	0.01691	0.18086	0.07363	6.020
1000	0.00544	0.02433	0.01008	0.05602	0.01144	0.12442	0.05073	18.998

Table 3 presents the Monte Carlo performance of the Ledoit–Wolf shrinkage estimator under different sample sizes. Compared with the SCM estimator, Ledoit–Wolf achieved lower Mean Squared Error (MSE) and Relative Frobenius Error (RFE), particularly for small and moderate sample sizes ($n = 25$, 50, and 100). As the sample size increased, the differences between SCM and Ledoit–Wolf gradually diminished, indicating convergence of both estimators. Although the shrinkage procedure required additional computation, the increase in runtime remained moderate relative to the improvement in estimation accuracy for smaller samples.
The Monte Carlo application results of the Oracle Approach Straining (OAS) covariance estimator (p = 5, R = 1000) are presented in Table 4.

Table 4. Monte Carlo performance of the Oracle Approximating Shrinkage (OAS) covariance estimator under the CSCN distribution (p = 5, R = 1000)

Sample Size (n)	Bias	Mean MSE	SD MSE	Mean RFE	SD RFE	Mean EE	SD EE	Runtime (s)
25	0.20996	0.86344	0.35622	0.33369	0.06805	1.05662	0.40390	0.443
50	0.12301	0.46689	0.20773	0.24489	0.05234	0.67086	0.26696	0.463
100	0.06499	0.24117	0.10280	0.17620	0.03669	0.44221	0.17651	0.486
500	0.01305	0.05051	0.02202	0.08061	0.01693	0.18313	0.07408	0.754
1000	0.00647	0.02435	0.01010	0.05603	0.01144	0.12517	0.05086	1.119

Table 4 summarizes the performance of the Oracle Approximating Shrinkage (OAS) estimator across different sample sizes. The OAS estimator consistently reduced the Mean Squared Error (MSE) and Relative Frobenius Error (RFE) compared with the Sample Covariance Matrix (SCM), particularly for smaller sample sizes. Its performance was comparable to that of the Ledoit–Wolf estimator, while requiring relatively low computational time. As the sample size increased, the estimation errors decreased steadily, demonstrating the asymptotic consistency of the OAS estimator under the CSCN distribution.
According to the obtained values, in small sample sizes (n = 25–100), OAS produces lower RFE and MSE compared to SCM. Ledoit–Wolf and OAS show very similar performance. In terms of runtime, OAS appears to be more advantageous than Ledoit–Wolf. For example, for n = 500, OAS ≈ 0.754 s, Ledoit–Wolf ≈ 6.020 s, and for n = 1000, OAS ≈ 1.119 s, Ledoit–Wolf ≈ 18.998 s were found.
The Monte Carlo application results of Tyler's M estimator (p = 5, R = 1000) are given in Table 5.

Table 5. Monte Carlo performance of Tyler's M-estimator under the CSCN distribution (p = 5, R = 1000)

Sample Size (n)	Bias	Mean MSE	SD MSE	Mean RFE	SD RFE	Mean EE	SD EE	Runtime (s)
25	0.01789	1.08537	0.41678	0.37512	0.07120	0.75424	0.32280	4.490
50	0.00881	0.53486	0.20253	0.26351	0.04902	0.51251	0.22781	13.420
100	0.00599	0.26289	0.09982	0.18464	0.03490	0.36857	0.16881	10.638
500	0.00258	0.05266	0.02019	0.08264	0.01561	0.16239	0.07167	77.937
1000	0.00124	0.02572	0.00994	0.05772	0.01109	0.11303	0.05208	181.085

Table 5 presents the Monte Carlo performance of Tyler's M-estimator under the CSCN distribution. The estimator exhibited a consistent reduction in Bias, Mean Squared Error (MSE), Relative Frobenius Error (RFE), and Eigenvalue Error (EE) as the sample size increased, indicating asymptotic consistency. However, compared with the shrinkage estimators, Tyler's estimator required substantially longer computation times due to its iterative fixed-point estimation procedure. Under the Gaussian CSCN model considered in this study, Tyler's estimator did not consistently outperform the shrinkage-based estimators in terms of estimation accuracy, although it maintained competitive eigenvalue estimation performance for larger sample sizes.

Based on these results, Bias decreases steadily. Mean MSE decreases at approximately 1/h rate. Mean RFE also decreases significantly as sample size increases. Mean EE drops to very low levels, especially in large samples. Runtime is significantly higher compared to the other three methods; this is due to the iterative fixed-point algorithm of the Tyler estimator.

A comparative summary of the covariance matrix estimators under the CSCN distribution (p = 5, R = 1000) is given in Figures 1-4 and Table 6.

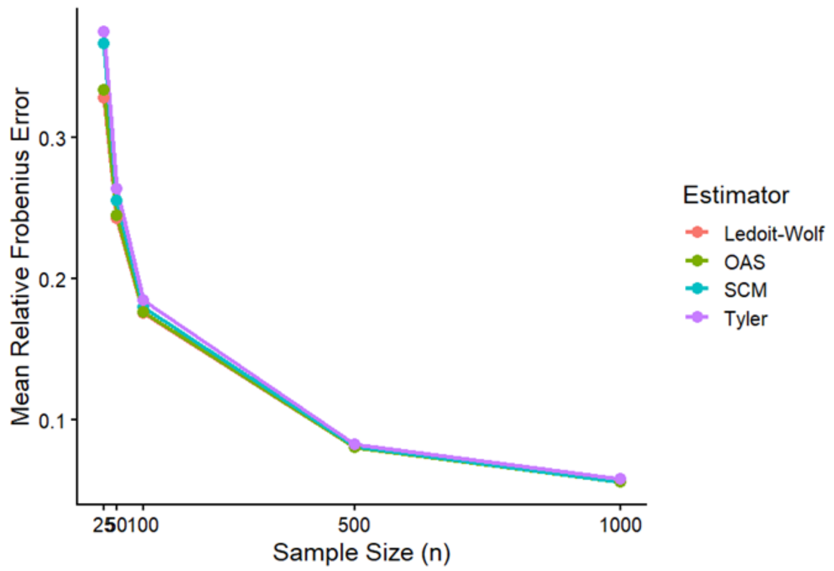

Figure 1. Mean RFE values

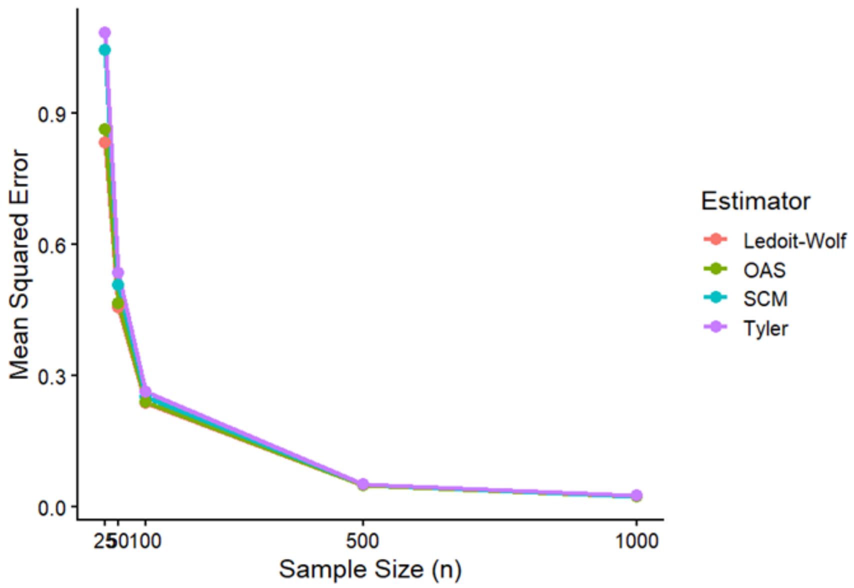

Figure 2. Values of mean squared error

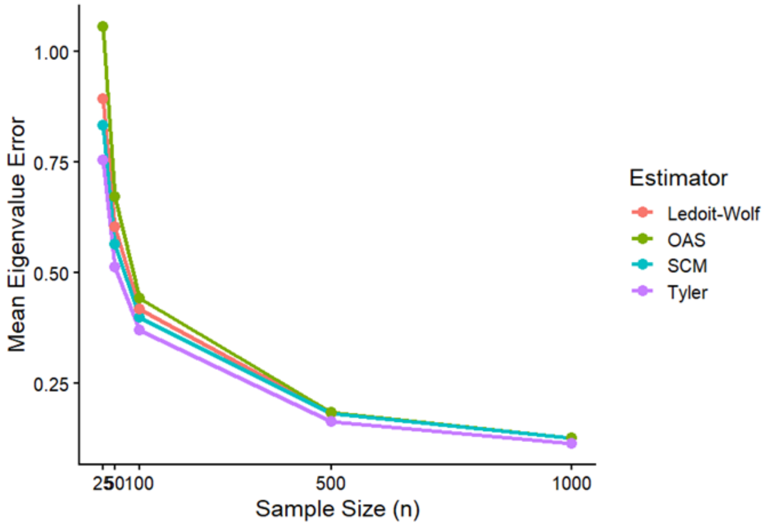

Figure 3. Values of mean eigenvalue error

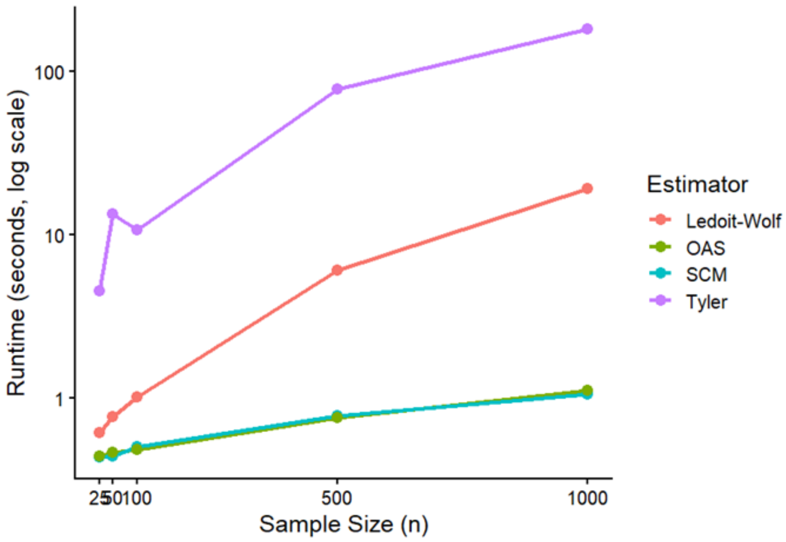

Figure 4. Values of runtime

Table 6. Comparative summary of covariance matrix estimators under the CSCN distribution (p = 5, R = 1000)

Estimator	n	Bias	Mean MSE	Mean RFE	Mean EE	Runtime (s)
SCM	25	0.01157	1.04508	0.36653	0.83228	0.436
Ledoit–Wolf	25	0.16615	**0.83333**	**0.32821**	0.89242	0.617
OAS	25	0.20996	0.86344	0.33369	1.05662	**0.443**
Tyler	25	0.01789	1.08537	0.37512	**0.75424**	4.490
SCM	50	0.00709	0.50707	0.25547	0.56399	**0.440**
Ledoit–Wolf	50	0.10197	**0.45736**	**0.24251**	0.60378	0.764
OAS	50	0.12301	0.46689	0.24489	0.67086	0.463
Tyler	50	0.00881	0.53486	0.26351	**0.51251**	13.420
SCM	100	0.00419	0.25169	0.17991	0.39898	0.500
Ledoit–Wolf	100	**0.05439**	**0.23874**	**0.17534**	0.41727	1.013
OAS	100	0.06499	0.24117	0.17620	0.44221	**0.486**
Tyler	100	0.00599	0.26289	0.18464	**0.36857**	10.638
SCM	500	0.00278	0.05117	0.08113	0.18034	0.777
Ledoit–Wolf	500	0.01100	**0.05044**	**0.08055**	0.18086	6.020
OAS	500	0.01305	0.05051	0.08061	0.18313	**0.754**
Tyler	500	0.00258	0.05266	0.08264	**0.16239**	77.937
SCM	1000	0.00129	0.02451	0.05622	0.12415	**1.049**
Ledoit–Wolf	1000	0.00544	**0.02433**	**0.05602**	0.12442	18.998
OAS	1000	0.00647	0.02435	0.05603	0.12517	1.119
Tyler	1000	0.00124	0.02572	0.05772	**0.11303**	181.085

SCM: Sample Covariance Matrix; **Ledoit–Wolf:**Ledoit–Wolf shrinkage estimator; **OAS:** Oracle Approximating Shrinkage estimator; **Tyler:** Tyler's M-estimator; **MSE:** Mean Squared Error; **RFE:** Relative Frobenius Error; **EE:** Eigenvalue Error. Results are based on 1000 Monte Carlo replications using a Hermitian Toeplitz covariance matrix under the Circularly Symmetric Complex Normal distribution.

As shown in Table 6, Ledoit–Wolf demonstrates the best performance in small and medium samples, particularly in terms of Mean MSE and Mean RFE. OAS offers very similar accuracy to Ledoit–Wolf, but stands out with its lower computation time, especially in large samples. Tyler's M-estimator provides the lowest error in terms of Mean EE (Eigenvalue Error), especially in large samples; however, its computation time is significantly higher due to its iterative nature. SCM is one of the simplest and fastest methods, and its performance approaches that of other methods as the sample size increases.

Overall, Ledoit–Wolf exhibited the best balance between estimation accuracy and computational efficiency, whereas Tyler's estimator provided the most accurate eigenvalue estimation at the expense of substantially increased computational time.

The Monte Carlo simulations revealed several consistent patterns across all covariance estimators. First, estimation accuracy improved monotonically with increasing sample size for all methods, confirming their asymptotic convergence under the CSCN distribution. Second, the Ledoit–Wolf and OAS shrinkage estimators consistently achieved lower Mean Squared Error (MSE) and Relative Frobenius Error (RFE) than the classical Sample Covariance Matrix (SCM), particularly when the sample size was limited. Third, Tyler's M-estimator provided the most accurate preservation of the covariance eigenstructure for larger sample sizes, although this improvement was accompanied by substantially higher computational cost due to its iterative estimation algorithm. Finally, the performancedifferences among the estimators became progressively smaller as the sample size increased, suggesting that all estimators converge toward the true covariance matrix in large-sample settings.

IV. Discussion

The Sample Covariance Matrix (SCM) exhibited satisfactory performance for large sample sizes; however, its estimation accuracy deteriorated when the number of observations was limited. This behavior has been extensively documented in multivariate statistical theory, where SCM is known to possess relatively high sampling variability under small-sample conditions. Consequently, eigenvalue dispersion increases and the covariance matrix may become poorly conditioned, particularly in moderate- and high-dimensional settings (Ledoit& Wolf, 2004; Goodman, 1963).

Among the evaluated estimators, the Ledoit–Wolf shrinkage estimator consistently achieved the lowest Mean Squared Error and Relative Frobenius Error for small and moderate sample sizes. The improvement arises from the shrinkage principle, whereby the sample covariance matrix is combined with a structured target matrix to reduce estimation variance while introducing only a limited amount of bias. This bias–variance trade-off has been shown theoretically to minimize expected Frobenius loss, particularly when the sample size is relatively small compared with the matrix dimension (Ledoit& Wolf, 2004).

V. Conclusion

This study presented a comprehensive Monte Carlo comparison of four Hermitian covariance matrix estimators under the Circularly Symmetric Complex Normal (CSCN) distribution. Using Hermitian positive-definite Toeplitz covariance matrices and multiple sample size scenarios, the estimators were assessed in terms of Bias, Mean Squared Error (MSE), Relative Frobenius Error (RFE), Eigenvalue Error (EE), and computational runtime.

The simulation results showed that all estimators exhibited improved performance as the sample size increased, confirming their asymptotic behavior. The Sample Covariance Matrix (SCM) remained computationally efficient and achieved increasingly accurate estimates for large samples. However, its estimation accuracy was generally inferior to that of shrinkage-based methods when the sample size was limited.

Among the evaluated methods, the Ledoit–Wolf shrinkage estimator consistently achieved the lowest MSE and Relative Frobenius Error in small and moderate sample size settings, demonstrating the effectiveness of shrinkage regularization. The Oracle Approximating Shrinkage (OAS) estimator produced performance comparable to Ledoit–Wolf while maintaining relatively low computational cost. Tyler's M-estimator showed competitive performance in preserving the eigenvalue structure of the covariance matrix, particularly for larger sample sizes, although its iterative estimation procedure resulted in substantially higher computational time.

Overall, the results indicate that shrinkage estimators provide the best balance between estimation accuracy and computational efficiency under the CSCN distribution, especially when sample sizes are limited. The proposed simulation framework also offers a reproducible benchmark for evaluating Hermitian covariance matrix estimators in complex-valued statistical analysis.

Future research may extend this framework to higher-dimensional covariance matrices, non-Gaussian complex distributions, heavy-tailed models, contaminated data, and high-dimensional settings where the number of variables approaches or exceeds the sample size.

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