International Journal of Engineering Inventions
ISSN: 2278-7461, ISBN: 2319-6491
Volume 1, Issue 12 (December 2012) PP: 75-87

Periodic Solution for Nonlinear System of Differential Equations with Pulse Action of Parameters

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Abstract:- In this paper we study the existence of a periodic solution for nonlinear system of differential equations with pulse action of parameters. The numerical-analytic method has been used to study the periodic solutions of the nonlinear ordinary differential equations that were introduced by Somioleko And the result of this study which is the using the periodic solutions on a wide range in difference processes in industry and technology.

I. INTRODUCTION

There are many subjects in physics and technology using mathematical methods that depends on the nonlinear differential equations, and it became clear that the existence of the periodic solutions and it's algorithm structure from more important problems in the present time. Where many of studies and researches dedicates for treatment the autonomous and non-autonomous periodic systems and specially with the integral equations and differential equations and the linear and nonlinear differential and which is dealing in general shape with the problems about periodic solutions theory and the modern methods in its quality treatment for the periodic differential equations.

Somioleko [6] assumes the numerical analytic method to study the periodic solutions for the ordinary differential equations and its algorithm structure and this method include uniformly sequences of the periodic functions and the result of that study is the using of the periodic solutions on a wide range for example see [4, 5, 6].

Consider the following system of nonlinear differential equation, which has the form:

$$\left.\begin{aligned} \frac{dx}{dt} &= \lambda x + f(t,x,y) \quad , \quad t \neq t_i \quad , \quad \Delta x \Big|_{t=t_i} = I_i(x,y) \\ \frac{dy}{dt} &= \beta x + g(t,x,y) \quad , \quad t \neq t_i \quad , \quad \Delta y \Big|_{t=t_i} = G_i(x,y) \end{aligned}\right\} \quad \ldots (1)$$

Where $x \in D_\lambda \subseteq R^n$, $y \in D_\beta \subseteq R^n$, D_λ is a closed and bounded domain.

The vector functions $f(t,x,y)$ and $g(t,x,y)$ are defined on the domain:

$$(t,x,y) \in R^1 \times D_\lambda \times D_\beta = (-\infty,\infty) \times D_\lambda \times D_\beta \quad \ldots (2)$$

Which are continuous in t, x, y and periodic in t of period T, where D_β is bounded domains subset of Euclidean spaces R^m , and the functions $I_i(x,y)$, $G_i(x,y)$ are continuous in the domain (2), where $I_{i+p}(x,y) = I_i(x,y)$, $G_{i+p}(x,y) = G_i(x,y)$ and $t_{i+p}(x,y) = t_i + T$ for p is a positive integer and $\{t_i\}$ is finite positive sequence of numbers.

Suppose that the vector functions in (1) are satisfying the following inequalities:

$$\max_{\substack{(x,y)\in D_\lambda \times D_\beta \\ t\in[0,T]}} \|f(t,x,y)\| \le M_1 \quad , \quad \max_{\substack{(x,y)\in D_\lambda \times D_\beta \\ t\in[0,T]}} \|g(t,x,y)\| \le M_2 \quad \ldots (3)$$

$$\max_{\substack{(x,y)\in D_\lambda \times D_\beta \\ 1\le i\le p}} \|I_i(x,y)\| \le M_3 \quad , \quad \max_{\substack{(x,y)\in D_\lambda \times D_\beta \\ 1\le i\le p}} \|G_i(x,y)\| \le M_4 \quad \ldots (4)$$

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| \quad \ldots (5)$$

$$\|g(t,x_1,y_1) - g(t,x_2,y_2)\| \le L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| \quad \ldots (6)$$

$$\|I_i(x_1,y_1) - I_i(x_2,y_2)\| \le K_3\|x_1 - x_2\| + K_4\|y_1 - y_2\| \quad \ldots (7)$$

$$\|G_i(x_1,y_1) - G_i(x_2,y_2)\| \le L_3\|x_1 - x_2\| + L_4\|y_1 - y_2\| \quad \ldots (8)$$

Where $t \in R'$, $x, x_1, x_2 \in D_\lambda$, $y, y_1, y_2 \in D_\beta$ and $M_1, M_2, M_3, M_4, K_1, K_2, K_3, K_4$, L_1, L_2, L_3, L_4 are a positive constant , $\|.\| = \max_{0\le t\le T} |.|$.

Let, β are a positive parameter defined in (2), continuous and periodic at τ, s, t and satisfy both following inequalities:

$$\left.\begin{array}{l} \left\|e^{\lambda(t-s)}\right\| \le H \\ \\ \left\|e^{\beta(t-s)}\right\| \le F \end{array}\right\} \qquad \ldots (9)$$

Where H and F are a positive constants.

We define the non-empty sets as follows:

$$\left.\begin{array}{l} D_{\lambda f} = D_{\lambda} - \left(\left(HT - \dfrac{\lambda T H^2 T}{1+e^{\lambda T}}\right) M_1 + pH\left(1 + \dfrac{\lambda T}{1+e^{\lambda T}}\right)\right) , \\ \\ D_{\beta f} = D_{\beta} - \left(\left(FT - \dfrac{\beta T F^2 T}{1+e^{\beta T}}\right) M_2 + pF\left(1 + \dfrac{\beta T}{1+e^{\beta T}}\right)\right) \end{array}\right\} \qquad \ldots (10)$$

Furthermore, we suppose that the larges Eigen-value for the following matrix:

$$\Lambda_0 = \begin{bmatrix} W_1 & W_2 \\ & \\ W_3 & W_4 \end{bmatrix}$$

is less than one, i.e. :

$$q_{max} = \frac{W_1 + W_4 + \sqrt{(W_1+W_4)^2 + 4(W_2W_3 - W_1W_4)}}{2} , \qquad \ldots (11)$$

where

$$W_1 = \left(HT - \frac{\lambda T H^2 T}{1+e^{\lambda T}}\right) K_1 + 2pH\left(1 + \frac{\lambda T}{1+e^{\lambda T}}\right) K_3 ,$$

$$W_2 = \left(HT - \frac{\lambda T H^2 T}{1+e^{\lambda T}}\right) K_2 + 2pH\left(1 + \frac{\lambda T}{1+e^{\lambda T}}\right) K_4 ,$$

$$W_3 = \left(FT - \frac{\beta T F^2 T}{1+e^{\beta T}}\right) L_1 + 2pF\left(1 + \frac{\beta T}{1+e^{\beta T}}\right) L_3 ,$$

$$W_4 = \left(FT - \frac{\beta T F^2 T}{1+e^{\beta}}\right) L_2 + 2pF\left(1 + \frac{\beta T}{1+e^{\beta}}\right) L_4 ,$$

$$M_5 = \left(HT - \frac{\lambda T H^2 T}{1+e^{\lambda T}}\right) M_1 + 2pH\left(1 + \frac{\lambda T}{1+e^{\lambda T}}\right) M_3 ,$$

$$M_6 = \left(FT - \frac{\beta T F^2 T}{1+e^{\beta T}}\right) M_2 + 2pF\left(1 + \frac{\beta T}{1+e^{\beta T}}\right) M_4 .$$

Approximation solution of (1)

The investigation of approximation solution of the system (1) will be introduced by the following theorem:

<u>Theorem 1</u>

If the system of nonlinear differential equations with pulse action (1) satisfy the inequalities (3) -- (8) and the conditions (9) , (10) then the sequences of functions :

$$\begin{aligned} x_{m+1}(t, x_o, y_o) = {} & x_o + \int_0^t e^{\lambda(t-s)}[f(s, x_m(s,x_o,y_o), y_m(s,x_o,y_o)) - \\ & - \int_0^T \frac{\lambda}{1+e^{\lambda T}} e^{\lambda(T-s)} f(s, x_m(s,x_o,y_o), y_m(s,x_o,y_o))ds]ds + \\ & + \sum_{0<t_i<t} e^{\lambda(T-s)} I_i(x_m(t,x_o,y_o), y_m(t,x_o,y_o)) - \\ & - \frac{\lambda}{1+e^{\lambda T}} \sum_{i=1}^{p} e^{\lambda(T-s)} I_i(x_m(t,x_o,y_o), y_m(t,x_o,y_o)) \qquad \ldots (12) \end{aligned}$$

with

$x_o(t, x_o) = x_o e^{\lambda t}$, *m =0 , 1 , 2 , ... ,*

end

$$y_{m+1}(t, x_o, y_o) = y_o + \int_0^t e^{\beta(t-s)}[f(s, x_m(s,x_o,y_o), y_m(s,x_o,y_o)) -$$

$$- \int_0^T \frac{\beta}{1+e^{\beta T}} e^{\beta(T-s)} f\big(s, x_m(s,x_o,y_o), y_m(s,x_o,y_o)\big) ds]ds +$$
$$+ \sum_{0<t_i<t} e^{\beta(T-s)} G_i\big(x_m(t,x_o,y_o), y_m(t,x_o,y_o)\big) -$$
$$- \frac{\beta}{1+e^{\beta T}} \sum_{i=1}^{p} e^{\beta(T-s)} G_i\big(x_m(t,x_o,y_o), y_m(t,x_o,y_o)\big) \qquad ... (13)$$

With

$y_o(t,x_o) = y_o e^{\beta t}$, $m = 0, 1, 2, \ldots\ldots,$

Are periodic in t of period T, and are uniformly convergent as $m \to \infty$ in the domain :

$$(t,x_o,y_o) \in R^{'} \times D_{\lambda f} \times D_{\beta f} \qquad ... (14)$$

To the limit function $x_\infty(t,x_o,y_o)$and$y_\infty(t,x_o,y_o)$ define in the domain (14), which is periodic in t of period T and satisfying the system of integral equations :

$$x(t,x_o,y_o) = x_o + \int_0^t e^{\lambda(t-s)}[f\big(s, x(s,x_o,y_o), y(s,x_o,y_o)\big) -$$
$$- \int_0^T \frac{\lambda}{1+e^{\lambda T}} e^{\lambda(T-s)} f\big(s, x(s,x_o,y_o), y(s,x_o,y_o)\big) ds]ds +$$
$$+ \sum_{0<t_i<t} e^{\lambda(T-s)} I_i\big(x(t,x_o,y_o), y(t,x_o,y_o)\big) -$$
$$- \frac{\lambda}{1+e^{\lambda T}} \sum_{i=1}^{p} e^{\lambda(T-s)} I_i\big(x(t,x_o,y_o), y(t,x_o,y_o)\big) , \qquad ... (15)$$

and

$$y(t,x_o,y_o) = y_o + \int_0^t e^{\beta(t-s)}[f\big(s, x(s,x_o,y_o), y(s,x_o,y_o)\big) -$$
$$- \int_0^T \frac{\beta}{1+e^{\beta T}} e^{\beta(T-s)} f\big(s, x(s,x_o,y_o), y(s,x_o,y_o)\big) ds]ds +$$
$$+ \sum_{0<t_i<t} e^{\beta(T-s)} G_i\big(x(t,x_o,y_o), y(t,x_o,y_o)\big) -$$
$$- \frac{\beta}{1+e^{\beta T}} \sum_{i=1}^{p} e^{\beta(T-s)} G_i\big(x(t,x_o,y_o), y(t,x_o,y_o)\big) , \qquad ... (16)$$

Which is are unique solutions of the system (1), provided that :

$$\|x^o(t,x_o,y_o) - x_o\| \le \left(HT - \frac{\lambda T H^2 T}{1+e^{\lambda T}}\right) M_1 + 2pH\left(1 + \frac{\lambda T}{1+e^{\lambda T}}\right) M_3 \qquad ... (17)$$
$$\|y^o(t,x_o,y_o) - y_o\| \le \left(FT - \frac{\beta T F^2 T}{1+e^{\beta T}}\right) M_2 + 2pF\left(1 + \frac{\beta T}{1+e^{\beta T}}\right) M_4 \qquad ... (18)$$
$$\begin{pmatrix} \|x_\infty(t,x_o,y_o) - x_m(t,x_o,y_o)\| \\ \\ \|y_\infty(t,x_o,y_o) - y_m(t,x_o,y_o)\| \end{pmatrix} \le \Lambda_o^m (E - \Lambda_o)^{-1} V_o , \qquad ... (19)$$

for all $t \in [0,T]$, $x_o \in D_{\lambda f}$, $y_o \in D_{\beta f}$, when:

$$\Lambda_o = \begin{pmatrix} N_1(t) & N_3(t) \\ \\ N_2(t) & N_4(t) \end{pmatrix}$$

And where

$$N_1(t) = \left(HT - \frac{\lambda T H^2 T}{1+e^{\lambda T}}\right) K_1 + 2pH\left(1 + \frac{\lambda T}{1+e^{\lambda T}}\right) K_3 , \qquad W_1 = \max_{t\in[0,T]} N_1(t)$$
$$N_2(t) = \left(HT - \frac{\lambda T H^2 T}{1+e^{\lambda T}}\right) K_2 + 2pH\left(1 + \frac{\lambda T}{1+e^{\lambda T}}\right) K_4 , \qquad W_2 = \max_{t\in[0,T]} N_2(t)$$

$$N_3(t)=\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_1+2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_3 \quad , \qquad W_3=\max_{t\in[0,T]}N_3(t)$$

$$N_4(t)=\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_2+2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_4 \quad , \qquad W_4=\max_{t\in[0,T]}N_4(t)$$

$$V_o=\begin{pmatrix}\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)M_1+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)M_3\\ \\ \left(FT-\frac{\beta TF^2T}{1+e^{\beta T}}\right)M_2+2pF\left(1+\frac{\beta T}{1+e^{\beta T}}\right)M_4\end{pmatrix}$$

Proof:

Setting m = 0 and using (12) , and the condition (9), we get

$$\|x_1(t,x_o,y_o)-x_o\|=\int_0^t\left\|e^{\lambda(t-s)}\right\|\|f(s,x_o,y_o)\|ds-$$

$$-\frac{\lambda t}{1+e^{\lambda T}}\int_0^T\left\|e^{2\lambda(T-s)}\right\|\|f(s,x_o,y_o)\|ds+\sum_{0<t_i<t}\left\|e^{\lambda(T-s)}\right\|\|I_i(x_o,y_o)\|-$$

$$-\frac{\lambda t}{1+e^{\lambda T}}\sum_{i=1}^{p}\left\|e^{\lambda(T-s)}\right\|\|I_i(x_o,y_o)\|$$

$$\le\left\|H-\frac{\lambda tH^2}{1+e^{\lambda T}}\right\|\int_0^t\|f(s,x_o,y_o)\|ds-\frac{\lambda tH^2}{1+e^{\lambda T}}\int_0^T\|f(s,x_o,y_o)\|ds+$$

$$+H\sum_{0<t_i<t}\|I_i(x_o,y_o)\|-\frac{\lambda tH}{1+e^{\lambda T}}\sum_{i=1}^{p}\|I_i(x_o,y_o)\|$$

Hence

$$\|x_1(t,x_o,y_o)-x_o\|\le\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)M_1+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)M_3 \qquad ...\ (19)$$

So that $x_1(t,x_o,y_o)\in D_\lambda$, for all $t\in R'$, $x_o\in D_{\lambda f}$,and by mathematic induction we get:

$$\|x_m(t,x_o,y_o)-x_o\|\le\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)M_1+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)M_3$$

and

$$\|y_1(t,x_o,y_o)-y_o\|\le\left(FT-\frac{\beta TF^2T}{1+e^{\beta T}}\right)M_2+2pF\left(1+\frac{\beta T}{1+e^{\beta T}}\right)M_4 \qquad ...\ (20)$$

Hence $y_1(t,x_o,y_o)\in D_\beta$, for all $t\in R'$, $y_o\in D_{\beta f}$. and by mathematic induction we get :

$$\|y_m(t,x_o,y_o)-y_o\|\le\left(FT-\frac{\beta TF^2T}{1+e^{\beta T}}\right)M_2+2pF\left(1+\frac{\beta T}{1+e^{\beta T}}\right)M_4$$

Then $x_m(t,x_o,y_o)\in D_\lambda$, $y_m(t,x_o,y_o)\in D_\beta$, $x_o\in D_{\lambda f}$, $y_o\in D_{\beta f}$.

We claim that the sequence of functions (12) and (13) are uniformly convergent on the domain (14). By using (19), and when m = 1 , we get

$$\|x_2(t,x_o,y_o)-x_1(t,x_o,y_o)\|=\int_0^t\left\|e^{\lambda(t-s)}\right\|\|f(s,x_1,y_1)\|ds-$$

$$-\frac{\lambda t}{1+e^{\lambda T}}\int_0^t\left\|e^{\lambda(T-s)}\right\|\|f(s,x_1,y_1)\|ds+\frac{\lambda t}{1+e^{\lambda T}}\int_0^T\left\|e^{\lambda(T-s)}\right\|\|f(s,x_1,y_1)\|ds+$$

$$+\sum_{0<t_i<t}\left\|e^{\lambda(T-s)}\right\|\|I_i(x_1,y_1)\|+\frac{\lambda t}{1+e^{\lambda T}}\sum_{i=1}^{p}\left\|e^{\lambda(T-s)}\right\|\|I_i(x_1,y_1)\|-$$

$$-\int_0^t\left\|e^{\lambda(t-s)}\right\|\|f(s,x_o,y_o)\|ds+\frac{\lambda t}{1+e^{\lambda T}}\int_0^t\left\|e^{\lambda(T-s)}\right\|\|f(s,x_o,y_o)\|ds+$$

$$+\ \frac{\lambda t}{1+e^{\lambda T}}\int_0^T \left\|e^{\lambda(T-s)}\right\|\|f(s,x_o,y_o)\|ds - \sum_{0<t_i<t}\left\|e^{\lambda(T-s)}\right\|\|I_i(x_o,y_o)\| -$$
$$-\ \frac{\lambda t}{1+e^{\lambda T}}\sum_{i=1}^{p}\left\|e^{\lambda(T-s)}\right\|\|I_i(x_o,y_o)\| \quad ,$$

then

$$\|x_2(t,x_o,y_o)-x_1(t,x_o,y_o)\| \le \left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_1+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_3\right]$$
$$\|x_1(t,x_o,y_o)-x_o\| +$$
$$+\left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_2+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_4\right]$$
$$\|y_1(t,x_o,y_o)-y_o\| \qquad ...\ (21)$$

And by mathematic induction we get :

$$\|x_{m+1}(t,x_o,y_o)-x_m(t,x_o,y_o)\| \le \left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_1+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_3\right]$$
$$\|x_m(t,x_o,y_o)-x_{m-1}(t,x_o,y_o)\| +$$
$$+\left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_2+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_4\right]$$
$$\|y_m(t,x_o,y_o)-y_{m-1}(t,x_o,y_o)\| \ , \qquad ...\ (22)$$

and

$$\|y_2(t,x_o,y_o)-y_1(t,x_o,y_o)\| \le \left[\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_1+2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_3\right]$$
$$\|x_1(t,x_o,y_o)-x_o\| +$$
$$+\left[\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_2+2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_4\right]$$
$$\|y_1(t,x_o,y_o)-y_o\| \qquad ...\ (23)$$

And by mathematic induction we get :

$$\|y_{m+1}(t,x_o,y_o)-y_m(t,x_o,y_o)\| \le \left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_1+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_3\right]$$
$$\|x_m(t,x_o,y_o)-x_{m-1}(t,x_o,y_o)\| +$$
$$+\left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_2+2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_4\right]$$
$$\|y_m(t,x_o,y_o)-y_{m-1}(t,x_o,y_o)\| \ , \qquad ...\ (24)$$

Rewrite inequalities (22) and (24) in vector form as :

$$V_{m+1}(t,x_o,y_o) \le \Lambda(t)V_m(t,x_o,y_o) \qquad ...\ (25)$$

where

$$V_{m+1}(t,x_o,y_o) = \begin{pmatrix}\|x_{m+1}(t,x_o,y_o)-x_m(t,x_o,y_o)\| \\ \\ \|y_{m+1}(t,x_o,y_o)-y_m(t,x_o,y_o)\|\end{pmatrix}$$

$$\Lambda(t) = \begin{pmatrix}N_1(t) & N_3(t) \\ \\ N_2(t) & N_4(t)\end{pmatrix}$$

and

$$V_m(t,x_o,y_o) = \begin{pmatrix}\|x_m(t,x_o,y_o)-x_{m-1}(t,x_o,y_o)\| \\ \\ \|y_m(t,x_o,y_o)-y_{m-1}(t,x_o,y_o)\|\end{pmatrix}$$

It follows form the inequality (25) that :

$$V_{m+1} \le \Lambda_o(t)V_m \qquad ...\ (26)$$

where

$$\Lambda_o = \max_{t\in[0,T]}|\Lambda(t)|$$

By iterating the inequality (3.20) , we find that

$$V_{m+1} \le \Lambda_o^m V_o$$

where

$$V_{\text{o}} = \begin{pmatrix} \left(HT - \dfrac{\lambda T H^2 T}{1+e^{\lambda T}}\right) M_1 + 2pH\left(1 + \dfrac{\lambda T}{1+e^{\lambda T}}\right) M_3 \\ \\ \left(FT - \dfrac{\beta T F^2 T}{1+e^{\beta T}}\right) M_2 + 2pF\left(1 + \dfrac{\beta T}{1+e^{\beta T}}\right) M_4 \end{pmatrix}$$

which leads to the estimate

$$\sum_{i=1}^{m} V_{\text{i}} \le \sum_{i=1}^{m} \Lambda_{\text{o}}^{i-1} V_{\text{o}} \qquad \ldots (27)$$

Since the matrix Λ_{o} has Eigen-values like as (12), then the series (27) is uniformly convergent, i.e.

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Lambda_{\text{o}}^{i-1} V_{\text{o}} = \sum_{i=1}^{\infty} \Lambda_{\text{o}}^{i-1} V_{\text{o}} = (E - \Lambda_{\text{o}})^{-1} V_{\text{o}} \qquad \ldots (28)$$

where E is a unity matrix.

The limiting relation (28) signifies a uniform convergent of the sequence $x_m(t, x_{\text{o}}, y_{\text{o}})andy_m(t, x_{\text{o}}, y_{\text{o}})$ in the domain (14).

Let

$$\left.\begin{aligned} \lim_{m\to\infty} x_m(t, x_{\text{o}}, y_{\text{o}}) &= x_\infty(t, x_{\text{o}}, y_{\text{o}}) \\ \lim_{m\to\infty} y_m(t, x_{\text{o}}, y_{\text{o}}) &= y_\infty(t, x_{\text{o}}, y_{\text{o}}) \end{aligned}\right\} \qquad \ldots (29)$$

Since the sequences of functions $x_m(t, x_{\text{o}}, y_{\text{o}})andy_m(t, x_{\text{o}}, y_{\text{o}})$ are periodic in t with period T, then the limiting of them are also periodic in t with period T, end thus $x_\infty(t, x_{\text{o}}, y_{\text{o}}) = x(t, x_{\text{o}}, y_{\text{o}})$, $y_\infty(t, x_{\text{o}}, y_{\text{o}}) = y(t, x_{\text{o}}, y_{\text{o}})$.

Also from (29) the following inequality

$$\begin{pmatrix} \|x_\infty(t, x_{\text{o}}, y_{\text{o}}) - x_m(t, x_{\text{o}}, y_{\text{o}})\| \\ \\ \|y_\infty(t, x_{\text{o}}, y_{\text{o}}) - y_m(t, x_{\text{o}}, y_{\text{o}})\| \end{pmatrix} \le \Lambda_{\text{o}}^{m} (E - \Lambda_{\text{o}})^{-1} V_{\text{o}} \quad , \qquad \ldots (30)$$

is hold for $m \ge 1$, and hence

$x_\infty(t, x_{\text{o}}, y_{\text{o}}) = x(t, x_{\text{o}}, y_{\text{o}})$, $y_\infty(t, x_{\text{o}}, y_{\text{o}}) = y(t, x_{\text{o}}, y_{\text{o}})$

which are the solutions of the system (1).

Uniqueness of Solution of (1)

Let all assumptions and conditions of theorem 1 were given, then the two functions $x(t, x_{\text{o}}, y_{\text{o}})$ and $y(t, x_{\text{o}}, y_{\text{o}})$are uniqueness of solution of (1) in the domain (14).

Proof :

Let

$$\begin{aligned} \hat{x}(t, x_{\text{o}}, y_{\text{o}}) = \; & x_{\text{o}} + \int_0^t e^{\lambda(t-s)} [f\big(s, \hat{x}(s, x_{\text{o}}, y_{\text{o}}), \hat{y}(s, x_{\text{o}}, y_{\text{o}})\big) - \\ & - \int_0^T \frac{\lambda}{1+e^{\lambda T}} e^{\lambda(T-s)} f\big(s, \hat{x}(s, x_{\text{o}}, y_{\text{o}}), \hat{y}(s, x_{\text{o}}, y_{\text{o}})\big) ds] ds + \\ & + \sum_{0<t_i<t} e^{\lambda(T-s)} I_i\big(\hat{x}(t, x_{\text{o}}, y_{\text{o}}), \hat{y}(t, x_{\text{o}}, y_{\text{o}})\big) - \\ & - \frac{\lambda}{1+e^{\lambda T}} \sum_{i=1}^{p} e^{\lambda(T-s)} I_i\big(\hat{x}(t, x_{\text{o}}, y_{\text{o}}), \hat{y}(t, x_{\text{o}}, y_{\text{o}})\big) \; , \end{aligned} \qquad \ldots (31)$$

and

$$\begin{aligned} \hat{y}(t, x_{\text{o}}, y_{\text{o}}) = \; & y_{\text{o}} + \int_0^t e^{\beta(t-s)} [f\big(s, \hat{x}(s, x_{\text{o}}, y_{\text{o}}), \hat{y}(s, x_{\text{o}}, y_{\text{o}})\big) - \\ & - \int_0^T \frac{\beta}{1+e^{\beta T}} e^{\beta(T-s)} f\big(s, \hat{x}(s, x_{\text{o}}, y_{\text{o}}), \hat{y}(s, x_{\text{o}}, y_{\text{o}})\big) ds] ds + \end{aligned}$$

$$+\sum_{0<t_i<t} e^{\beta(T-s)}G_i\big(\hat{x}(t,x_o,y_o),\hat{y}(t,x_o,y_o)\big) -$$
$$-\frac{\beta}{1+e^{\beta T}}\sum_{i=1}^{p} e^{\beta(T-s)}G_i\big(\hat{x}(t,x_o,y_o),\hat{y}(t,x_o,y_o)\big) \quad , \qquad ... (32)$$

Are another solutions for the system (1), then we shall prove that $x(t,x_o,y_o)=\hat{x}(t,x_o,y_o)$, $(t,x_o,y_o)=\hat{y}(t,x_o,y_o)$, and to do this we need to prove the following inequality by induction,

$$\begin{pmatrix}\|\hat{x}(t,x_o,y_o)-x_m(t,x_o,y_o)\| \\ \\ \|\hat{y}(t,x_o,y_o)-y_m(t,x_o,y_o)\|\end{pmatrix} \le \Lambda_o^m \begin{pmatrix}\|\hat{x}(t,x_o,y_o)-x_{m-1}(t,x_o,y_o)\| \\ \\ \|\hat{y}(t,x_o,y_o)-y_{m-1}(t,x_o,y_o)\|\end{pmatrix} \quad ... (33)$$

for $m \ge 1$, where

$$M_1^* = \max_{\substack{(x,y)\in D_\lambda\times D_\beta \\ t\in[0,T]}} \|f(t,x,y)\| \quad , \quad M_2^* = \max_{\substack{(x,y)\in D_\lambda\times D_\beta \\ t\in[0,T]}} \|g(t,x,y)\|$$
$$M_3^* = \max_{\substack{(x,y)\in D_\lambda\times D_\beta \\ 1\le i\le p}} \|I_i(x,y)\| \quad , \quad M_4^* = \max_{\substack{(x,y)\in D_\lambda\times D_\beta \\ 1\le i\le p}} \|G_i(x,y)\|$$

For m = 0 in (15) and (16), we have

$$\|\hat{x}(t,x_o,y_o)-x_o\| \le \left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)M_1^* + 2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)M_3^* \qquad ... (34)$$

and

$$\|\hat{y}(t,x_o,y_o)-y_o\| \le \left(FT-\frac{\beta TF^2T}{1+e^{\beta T}}\right)M_2^* + 2pF\left(1+\frac{\beta T}{1+e^{\beta T}}\right)M_4^* \qquad ... (35)$$

and for m = 1, we get also

$$\|\hat{x}(t,x_o,y_o)-x_1(t,x_o,y_o)\| \le \left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_1 + 2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_3\right]$$
$$\|\hat{x}(t,x_o,y_o)-x_o\| +$$
$$+\left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_2 + 2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_4\right]$$
$$\|\hat{y}(t,x_o,y_o)-y_o\| \qquad ... (36)$$

and

$$\|\hat{y}(t,x_o,y_o)-y_1(t,x_o,y_o)\| \le \left[\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_1 + 2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_3\right]$$
$$\|\hat{x}(t,x_o,y_o)-x_o\| +$$
$$+\left[\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_2 + 2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_4\right]$$
$$\|\hat{y}(t,x_o,y_o)-y_o\| \qquad ... (37)$$

Suppose that (33) is true for m = p -1 , i.e.

$$\begin{pmatrix}\left\|\hat{x}(t,x_o,y_o)-x_{p-1}(t,x_o,y_o)\right\| \\ \\ \left\|\hat{y}(t,x_o,y_o)-y_{p-1}(t,x_o,y_o)\right\|\end{pmatrix} \le \Lambda_o^{p-1} \begin{pmatrix}\left\|\hat{x}(t,x_o,y_o)-x_{p-2}(t,x_o,y_o)\right\| \\ \\ \left\|\hat{y}(t,x_o,y_o)-y_{p-2}(t,x_o,y_o)\right\|\end{pmatrix} ... (38)$$

then

$$\left\|\hat{x}(t,x_o,y_o)-x_p(t,x_o,y_o)\right\| \le \left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_1 + 2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_3\right]$$
$$\left\|\hat{x}(t,x_o,y_o)-x_{p-1}(t,x_o,y_o)\right\| +$$
$$+\left[\left(HT-\frac{\lambda TH^2T}{1+e^{\lambda T}}\right)K_2 + 2pH\left(1+\frac{\lambda T}{1+e^{\lambda T}}\right)K_4\right]$$
$$\left\|\hat{y}(t,x_o,y_o)-y_{p-1}(t,x_o,y_o)\right\|$$

and

$$\left\|\hat{y}(t,x_o,y_o)-y_p(t,x_o,y_o)\right\| \le \left[\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_1 + 2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_3\right]$$
$$\left\|\hat{x}(t,x_o,y_o)-x_{p-1}(t,x_o,y_o)\right\| +$$
$$+\left[\left(FT-\frac{\beta TF^2T}{1+e^{\lambda T}}\right)L_2 + 2pF\left(1+\frac{\beta T}{1+e^{\lambda T}}\right)L_4\right]$$

$\|\hat{y}(t,x_o,y_o)-y_{p-1}(t,x_o,y_o)\|$

i.e.

$$\begin{pmatrix}\|\hat{x}(t,x_o,y_o)-x_p(t,x_o,y_o)\| \\ \\ \|\hat{y}(t,x_o,y_o)-y_p(t,x_o,y_o)\|\end{pmatrix} \le \Lambda_o^p \begin{pmatrix}\|\hat{x}(t,x_o,y_o)-x_{p-1}(t,x_o,y_o)\| \\ \\ \|\hat{y}(t,x_o,y_o)-y_{p-1}(t,x_o,y_o)\|\end{pmatrix}$$

Then the inequality (33) is true for m = 0, 1, 2, . . .

By iterating the inequality (33) gives :

$$\begin{pmatrix}\|\hat{x}(t,x_o,y_o)-x_m(t,x_o,y_o)\| \\ \\ \|\hat{y}(t,x_o,y_o)-y_m(t,x_o,y_o)\|\end{pmatrix} \le \Lambda_o^m \begin{pmatrix}\|\hat{x}(t,x_o,y_o)-x_{m-1}(t,x_o,y_o)\| \\ \\ \|\hat{y}(t,x_o,y_o)-y_{m-1}(t,x_o,y_o)\|\end{pmatrix}$$

But from the condition (11) we obtain $\Lambda_o^m \to 0$ as $m \to \infty$, hence, proceeding in the last inequality to the limit we obtain that $x(t,x_o,y_o)=\hat{x}(t,x_o,y_o)$ and $y(t,x_o,y_o)=\hat{y}(t,x_o,y_o)$ which proves that the two solutions $x(t,x_o,y_o)$ and $y(t,x_o,y_o)$ are unique, and this completes the proof of theorem 2.

Existence of solution of (1)

The problem of existence solution of the system (1) is uniquely connected with the existence of zero of the function $\Delta(0,x_o,y_o)$ and $\Delta^*(0,x_o,y_o)$, which has the form:

$$\Delta(0,x_o,y_o)=\frac{\lambda}{1+e^{\lambda T}}[\int_0^T e^{\lambda(T-s)} f\big(t,x_\infty(t,x_o,y_o),y_\infty(t,x_o,y_o)\big)dt + \\ + \sum_{i=1}^{p} e^{\lambda(T-s)} I_i\big(x_\infty(t_i,x_o,y_o),y_\infty(t_i,x_o,y_o)\big)] \quad ...(39)$$

$$\Delta^*(0,x_o,y_o)=\frac{\beta}{1+e^{\beta T}}[\int_0^T e^{\beta(T-s)} f\big(t,x_\infty(t,x_o,y_o),y_\infty(t,x_o,y_o)\big)dt + \\ + \sum_{i=1}^{p} e^{\beta(T-s)} G_i\big(x_\infty(t_i,x_o,y_o),y_\infty(t_i,x_o,y_o)\big)] \quad ...(40)$$

Where $x_\infty(t,x_o,y_o)$ and $y_\infty(t,x_o,y_o)$ the sequence's limiting (12) and (13) successively, and this function is approximately determined from the sequence of functions:

$$\Delta_m(0,x_o,y_o)=\frac{\lambda}{1+e^{\lambda T}}[\int_0^T e^{\lambda(T-s)} f\big(t,x_m(t,x_o,y_o),y_m(t,x_o,y_o)\big)dt + \\ + \sum_{i=1}^{p} e^{\lambda(T-s)} I_i\big(x_m(t_i,x_o,y_o),y_m(t_i,x_o,y_o)\big)] \quad ...(41)$$

$$\Delta_m^*(0,x_o,y_o)=\frac{\beta}{1+e^{\beta T}}[\int_0^T e^{\beta(T-s)} f\big(t,x_m(t,x_o,y_o),y_m(t,x_o,y_o)\big)dt + \\ + \sum_{i=1}^{p} e^{\beta(T-s)} G_i\big(x_m(t_i,x_o,y_o),y_m(t_i,x_o,y_o)\big)] \quad ...(42)$$

where $m = 0, 1, 2, ...$

<u>Theorem 3 :</u>

Let all assumptions and conditions of theorem 1 were given, then the following inequality :

$$\begin{pmatrix}\|\Delta(0,x_o,y_o)-\Delta_m(0,x_o,y_o)\| \\ \\ \|\Delta^*(t,x_o,y_o)-\Delta_m^*(0,x_o,y_o\|\end{pmatrix} \le Q\,\Lambda_o^m(E-\Lambda_o)^{-1}V_o \quad ...(43)$$

where

$$Q=\begin{pmatrix}\frac{\lambda HT}{1+e^{\lambda T}}K_1+\frac{\lambda pH}{1+e^{\lambda T}}K_3 & \frac{\lambda HT}{1+e^{\lambda T}}K_2+\frac{\lambda pH}{1+e^{\lambda T}}K_4 \\ \\ \frac{\beta FT}{1+e^{\beta T}}L_1+\frac{\beta pF}{1+e^{\beta T}}L_3 & \frac{\beta FT}{1+e^{\beta T}}L_2+\frac{\beta pF}{1+e^{\beta T}}L_4\end{pmatrix}$$

Is holds for $m \ge 0$, $t \in [0,T]$, $x_o \in D_{\lambda f}$, $y_o \in D_{\beta f}$.

<u>proof</u> :

According to (39) and (41) , we have.

$$\| \Delta(0,x_o,y_o) - \Delta_m(0,x_o,y_o)\| \leq \frac{\lambda}{1+e^{\lambda T}}\int_0^T \left\|e^{\lambda(T-s)}\right\| \left\|f\big(t,x_\infty(t,x_o,y_o),y_\infty(t,x_o,y_o)\big) - \right.$$

$$\left. -f\big(t,x_m(t,x_o,y_o),y_m(t,x_o,y_o)\big)\right\| dt +$$

$$+\frac{\lambda}{1+e^{\lambda T}}\sum_{i=1}^{p}\left\|e^{\lambda(T-s)}\right\| \left\|I_i\big(x_\infty(t_i,x_o,y_o),y_\infty(t_i,x_o,y_o)\big) - \right.$$

$$\left. - I_i\big(x_m(t_i,x_o,y_o),y_m(t_i,x_o,y_o)\big)\right\| \leq$$

$$\leq \frac{\lambda HT}{1+e^{\lambda T}}\left[K_1\|x_\infty(t,x_o,y_o) - x_m(t,x_o,y_o)\| + K_2\|y_\infty(t,x_o,y_o) - y_m(t,x_o,y_o)\|\right] +$$

$$+ \frac{\lambda pH}{1+e^{\lambda T}}\left[K_3\|x_\infty(t,x_o,y_o) - x_m(t,x_o,y_o)\| + K_4\|y_\infty(t,x_o,y_o) - y_m(t,x_o,y_o)\|\right]$$

So that

$$\| \Delta(0,x_o,y_o) - \Delta_m(0,x_o,y_o)\| \leq \left(\frac{\lambda HT}{1+e^{\lambda T}}K_1 + \frac{\lambda pH}{1+e^{\lambda T}}K_3\right)\|x_\infty(t,x_o,y_o) - x_m(t,x_o,y_o)\|$$
$$+\left(\frac{\lambda HT}{1+e^{\lambda T}}K_2 + \frac{\lambda pH}{1+e^{\lambda T}}K_4\right)\|y_\infty(t,x_o,y_o) - y_m(t,x_o,y_o)\| \quad ...\ (44)$$

By the same method and by (40) and (42) , we have

$$\|\Delta^*(t,x_o,y_o) - \Delta_m^*(0,x_o,y_o\| \leq \left(\frac{\beta FT}{1+e^{\beta T}}L_1 + \frac{\beta pF}{1+e^{\beta T}}L_3\right)\|x_\infty(t,x_o,y_o) - x_m(t,x_o,y_o)\|$$
$$+\left(\frac{\beta FT}{1+e^{\beta T}}L_2 + \frac{\beta pF}{1+e^{\beta T}}L_4\right)\|y_\infty(t,x_o,y_o) - y_m(t,x_o,y_o)\| \quad ...\ (45)$$

And so on, rewrite the inequalities (44) and (45) in vector form as :

$$\begin{pmatrix}\| \Delta(0,x_o,y_o) - \Delta_m(0,x_o,y_o)\| \\ \\ \|\Delta^*(t,x_o,y_o) - \Delta_m^*(0,x_o,y_o\|\end{pmatrix} \leq Q\begin{pmatrix}\|x_\infty(t,x_o,y_o) - x_m(t,x_o,y_o)\| \\ \\ \|y_\infty(t,x_o,y_o) - y_m(t,x_o,y_o)\|\end{pmatrix}$$

And by (30) , we get

$$\begin{pmatrix}\| \Delta(0,x_o,y_o) - \Delta_m(0,x_o,y_o)\| \\ \\ \|\Delta^*(t,x_o,y_o) - \Delta_m^*(0,x_o,y_o\|\end{pmatrix} \leq Q\,\Lambda_o^m(E-\Lambda_o)^{-1}V_o$$

Theorem 4 :

Let thefunction $f(t,x,y)$ and $g(t,x,y)$ in the system (1) are defined on the intervals [a, b]and [c, d] respectively, and periodic in *t* with period *T*.

Let that the sequence of functions (41) satisfying the next inequalities :

$$\left.\begin{array}{l} \min\limits_{a+M_5\ \leq x_o \leq b-M_5} \Delta_m(0,x_o,y_o) \ \leq\ -\delta_m \\ \\ \max\limits_{a+M_5\ \leq x_o \leq b-M_5} \Delta_m(0,x_o,y_o) \ \geq\ \delta_m \end{array}\right\} \quad ...\ (46)$$

Let that the sequence of functions (42) satisfying the next inequalities :

$$\left.\begin{array}{l} \min\limits_{c+M_6\ \leq y_o \leq d-M_6} \Delta_m^*(0,x_o,y_o) \ \leq\ -\varepsilon_m \\ \\ \max\limits_{c+M_6\ \leq y_o \leq d-M_6} \Delta_m^*(0,x_o,y_o) \ \geq\ \varepsilon_m \ , \end{array}\right\} \quad ...\ (47)$$

for $m \geq 0$ where :

$$\delta_m = \left[\left(\frac{\lambda HT}{1+e^{\lambda T}}K_1 + \frac{\lambda pH}{1+e^{\lambda T}}K_3\right) + \left(\frac{\lambda HT}{1+e^{\lambda T}}K_2 + \frac{\lambda pH}{1+e^{\lambda T}}K_4\right)\right]\Lambda_o^m(E-\Lambda_o)^{-1}M_5\ ,$$

$$\varepsilon_m = \left[\left(\frac{\beta FT}{1+e^{\beta T}}L_1 + \frac{\beta pF}{1+e^{\beta T}}L_3\right) + \left(\frac{\beta FT}{1+e^{\beta T}}L_2 + \frac{\beta pF}{1+e^{\beta T}}L_4\right)\right]\Lambda_o^m(E-\Lambda_o)^{-1}M_6$$

Then the system (1) has periodic solution of period T, $x = x(t,x_o,y_o)$and $y = y(t,x_o,y_o)$for which $a+M_5 \leq x_o \leq b-M_5$, $c+M_6 \leq y_o \leq d-M_6$.

Proof:

Let x_1 , x_2 be any two points in the interval $[a+M_5, b-M_5]$ such that :

$$\left.\begin{array}{l}\Delta_m(0,x_1,y_1) = \min\limits_{a+M_5 \le x_o \le b-M_5} \Delta_m(0,x_o,y_o) \\ \\ \Delta_m(0,x_2,y_2) = \max\limits_{a+M_5 \le x_o \le b-M_5} \Delta_m(0,x_o,y_o)\end{array}\right\} \quad \ldots (48)$$

Let y_1 , y_2 be any two points in the interval $[c+M_6, d-M_6]$ such that :

$$\left.\begin{array}{l}\Delta_m^*(0,x_1,y_1) = \min\limits_{c+M_6 \le y_o \le d-M_6} \Delta_m^*(0,x_o,y_o) \\ \\ \Delta_m^*(0,x_2,y_2) = \max\limits_{c+M_6 \le y_o \le d-M_6} \Delta_m^*(0,x_o,y_o)\end{array}\right\} \quad \ldots (49)$$

By (43) and (46), we get :

$$\left.\begin{array}{l}\Delta(0,x_1,y_1) = \Delta_m(0,x_1,y_1) + [\Delta(0,x_1,y_1) - \Delta_m(0,x_1,y_1)] < 0 \\ \\ \Delta(0,x_2,y_2) = \Delta_m(0,x_2,y_2) + [\Delta(0,x_2,y_2) - \Delta_m(0,x_2,y_2)] > 0\end{array}\right\} \quad \ldots (50)$$

From (43) and (47), we get :

$$\left.\begin{array}{l}\Delta^*(0,x_1,y_1) = \Delta_m^*(0,x_1,y_1) + [\Delta^*(0,x_1,y_1) - \Delta_m^*(0,x_1,y_1)] < 0 \\ \\ \Delta^*(0,x_2,y_2) = \Delta_m^*(0,x_2,y_2) + [\Delta^*(0,x_2,y_2) - \Delta_m^*(0,x_2,y_2)] > 0\end{array}\right\} \quad \ldots (51)$$

It follows from the functions (39) , (40) and the relations (50) , (51) in virtue of the continuity of the Δ-constant, that there exists $x_\infty = x_o$, $x_\infty \in [x_1, x_2]$ and $y_\infty = y_o$, $y_\infty \in [y_1, y_2]$ such that $\Delta(0, x_\infty, y_\infty) = 0$, $\Delta^*(0, x_\infty, y_\infty) = 0$. And this proved that the system (1) has a periodic solution $x = x(t, x_o, y_o)$for $x_o \in [a + M_5, b - M_5]$and $y = y(t, x_o, y_o)$for $y_o \in [c + M_6, d - M_6]$.

Remark 1 [6] :

When $R^n = R'$, i.e. when x is a scalar theorem 4can be strengthens by giving up the requirement that the singular point shout be isolated, thus we have

Theorem 5 :

Let the function $\Delta(0, x_o, y_o)$ defined as$\Delta : D_{\lambda f} \to R'$

$$\Delta(0,x_o,y_o) = \frac{\lambda}{1+e^{\lambda T}} [\int_0^T e^{\lambda(T-s)} f\big(t, x_\infty(t,x_o,y_o), y_\infty(t,x_o,y_o)\big) dt + $$
$$+ \sum_{i=1}^{p} e^{\lambda(T-s)} I_i\big(x_\infty(t_i,x_o,y_o), y_\infty(t_i,x_o,y_o)\big)] \quad \ldots (52)$$

Let the function $\Delta^*(0, x_o, y_o)$ defined as$\Delta^* : D_{\beta f} \to R'$

$$\Delta^*(0,x_o,y_o) = \frac{\beta}{1+e^{\beta T}} [\int_0^T e^{\beta(T-s)} f\big(t, x_\infty(t,x_o,y_o), y_\infty(t,x_o,y_o)\big) dt + $$
$$+ \sum_{i=1}^{p} e^{\beta(T-s)} G_i\big(x_\infty(t_i,x_o,y_o), y_\infty(t_i,x_o,y_o)\big)] \quad \ldots (53)$$

Where the functions $x_\infty(t, x_o, y_o)$and$y_\infty(t, x_o, y_o)$are the limit of asequence of periodic functions (12) , (13) respectively, then the following inequalities are holds :

$$\|\Delta(0,x_o,y_o)\| \le \left(Ht - \frac{\lambda t H^2 T}{1+e^{\lambda T}}\right) M_1 + 2pH\left(1 + \frac{\lambda t}{1+e^{\lambda T}}\right) M_3 \quad \ldots (54)$$

$$\|\Delta^*(0,x_o,y_o)\| \le \left(Ft - \frac{\beta t F^2 T}{1+e^{\beta T}}\right) M_2 + 2pF\left(1 + \frac{\beta t}{1+e^{\beta T}}\right) M_4 \quad \ldots (55)$$

$$\begin{array}{rl}\|\Delta(0,x_o^1,y_o^1) - \Delta(0,x_o^2,y_o^2)\| \le & \overline{W_1}(1 - \overline{W_1} - \overline{W_2}\overline{W_3}(1-\overline{W_4})^{-1})^{-1} [\|x_o^1 - x_o^2\| + \\ & +\overline{W_2}(1-\overline{W_4})^{-1}\|y_o^1 - y_o^2\|] + \\ & +\overline{W_2}(1 - \overline{W_4} - \overline{W_2}\overline{W_3}(1-\overline{W_1})^{-1})^{-1} [\|y_o^1 - y_o^2\| + \\ & +\overline{\overline{W_3}}(1-\overline{W_1})^{-1}\|x_o^1 - x_o^2\|] \quad \ldots (56)\end{array}$$

$$\begin{array}{rl}\|\Delta^*(0,x_o^1,y_o^1) - \Delta^*(0,x_o^2,y_o^2)\| \le & \overline{W_3}(1 - \overline{W_1} - \overline{W_2}\overline{W_3}(1-\overline{W_4})^{-1})^{-1} [\|x_o^1 - x_o^2\| + \\ & +\overline{W_2}(1-\overline{W_4})^{-1}\|y_o^1 - y_o^2\|] + \\ & +\overline{W_4}(1 - \overline{W_4} - \overline{W_2}\overline{W_3}(1-\overline{W_1})^{-1})^{-1} [\|y_o^1 - y_o^2\| + \\ & +\overline{W_3}(1-\overline{W_1})^{-1}\|x_o^1 - x_o^2\|] \quad \ldots (57)\end{array}$$

Where

$\overline{W_1} = HT\left[\left(K_1 - \frac{\lambda T K_1}{1+e^{\lambda T}}\right) + \left(K_3 + \frac{\lambda K_3 p}{1+e^{\lambda T}}\right)\right]$,
$\overline{W_2} = HT\left[\left(K_2 - \frac{\lambda T K_2}{1+e^{\lambda T}}\right) + \left(K_4 + \frac{\lambda K_4 p}{1+e^{\lambda T}}\right)\right]$,
$\overline{W_3} = FT\left[\left(L_1 - \frac{\beta T L_1}{1+e^{\beta T}}\right) + \left(L_3 + \frac{\beta L_3 p}{1+e^{\beta T}}\right)\right]$,
$\overline{W_4} = FT\left[\left(L_2 - \frac{\beta T L_2}{1+e^{\beta T}}\right) + \left(L_4 + \frac{\beta L_4 p}{1+e^{\beta T}}\right)\right]$,
for all $x_o, x_o^1, x_o^2 \in D_{\lambda f}$ and $y_o, y_o^1, y_o^2 \in D_{\beta f}$.

Proof:

From the properties of the function $x_\infty(t,x_o,y_o)$ and $y_\infty(t,x_o,y_o)$ established by theorem 1, it follows that the functions $\Delta = \Delta(0,x_o,y_o)$ and $\Delta^* = \Delta^*(0,x_o,y_o)$ are continuous and bounded with the positive constants

$\left(Ht - \frac{\lambda t H^2 T}{1+e^{\lambda T}}\right) M_1 + 2pH\left(1 + \frac{\lambda t}{1+e^{\lambda T}}\right) M_3$ for $x_o \in D_{\lambda f}$, and

$\left(Ft - \frac{\beta t F^2 T}{1+e^{\beta T}}\right) M_2 + 2pF\left(1 + \frac{\beta t}{1+e^{\beta T}}\right) M_4$ for $y_o \in D_{\beta f}$, respectively.

By using (52) , we have

$$\begin{aligned}
\|\Delta(0,x_o^1,y_o^1) - \Delta(0,x_o^2,y_o^2)\| \le{}& \frac{\lambda}{1+e^{\lambda T}} \int_0^T \|e^{\lambda(T-s)}\| \|f(t,x_\infty(t,x_o^1,y_o^1),y_\infty(t,x_o^1,y_o^1)) - \\
&-f(t,x_\infty(t,x_o^2,y_o^2),y_\infty(t,x_o^2,y_o^2))\| dt + \\
&+\frac{\lambda}{1+e^{\lambda T}} \sum_{i=1}^{p} \|e^{\lambda(T-s)}\| \|I_i(x_\infty(t_i,x_o^1,y_o^1),y_\infty(t_i,x_o^2,y_o^2)) - \\
- I_i(x_\infty(t_i,x_o^1,y_o^1),y_\infty(t_i,x_o^2,y_o^2))\| \le \\
\le{}& \frac{\lambda HT}{1+e^{\lambda T}}[K_1\|x_\infty(t,x_o^1,y_o^1) - x_\infty(t,x_o^2,y_o^2)\| + K_2\|y_\infty(t,x_o^1,y_o^1) - y_\infty(t,x_o^2,y_o^2)\|] + \\
&+ \frac{\lambda pH}{1+e^{\lambda T}}[K_3\|x_\infty(t,x_o^1,y_o^1) - x_\infty(t,x_o^2,y_o^2)\| + K_4\|y_\infty(t,x_o^1,y_o^1) - y_\infty(t,x_o^2,y_o^2)\|]
\end{aligned}$$

So that

$$\begin{aligned}
\|\Delta(0,x_o^1,y_o^1) - \Delta(0,x_o^2,y_o^2)\| \le{}& \frac{\lambda H}{1+e^{\lambda T}}[(TK_1 + pK_3)\|x_\infty(t,x_o^1,y_o^1) - x_\infty(t,x_o^2,y_o^2)\| + \\
&+(TK_2 + pK_4)\|y_\infty(t,x_o^1,y_o^1) - y_\infty(t,x_o^2,y_o^2)\|] \quad ... \ (58)
\end{aligned}$$

And we will find by the same method and by (53) , we have :

$$\begin{aligned}
\|\Delta^*(0,x_o^1,y_o^1) - \Delta^*(0,x_o^2,y_o^2)\| \le{}& \frac{\beta F}{1+e^{\lambda T}}[(TL_1 + pL_3)\|x_\infty(t,x_o^1,y_o^1) - x_\infty(t,x_o^2,y_o^2)\| + \\
&+(TL_2 + pL_4)\|y_\infty(t,x_o^1,y_o^1) - y_\infty(t,x_o^2,y_o^2)\|] \quad ... \ (59)
\end{aligned}$$

Where $x_\infty(t,x_o^1,y_o^1)$, $x_\infty(t,x_o^2,y_o^2)$ and $y_\infty(t,x_o^1,y_o^1)$, $y_\infty(t,x_o^2,y_o^2)$ are the solutions of the following integral equations :

$$\begin{aligned}
x(t,x_o^k,y_o^k) ={}& x_o^k + \int_0^t e^{\lambda(t-s)}[f(s,x(s,x_o^k,y_o^k),y(s,x_o^k,y_o^k)) - \\
&- \int_0^T \frac{\lambda}{1+e^{\lambda T}} e^{\lambda(T-s)} f(s,x(s,x_o^k,y_o^k),y(s,x_o^k,y_o^k))ds]ds + \\
&+ \sum_{0<t_i<t} e^{\lambda(T-s)} I_i(x(t_i,x_o^k,y_o^k),y(t_i,x_o^k,y_o^k)) - \\
&- \frac{\lambda}{1+e^{\lambda T}} \sum_{i=1}^{p} e^{\lambda(T-s)} I_i(x(t_i,x_o^k,y_o^k),y(t_i,x_o^k,y_o^k)) \ , \quad ... \ (60)
\end{aligned}$$

And

$$\begin{aligned}
y(t,x_o^k,y_o^k) ={}& y_o^k + \int_0^t e^{\beta(t-s)}[f(s,x(s,x_o^k,y_o^k),y(s,x_o^k,y_o^k)) - \\
&- \int_0^T \frac{\beta}{1+e^{\beta T}} e^{\beta(T-s)} f(s,x(s,x_o^k,y_o^k),y(s,x_o^k,y_o^k))ds]ds +
\end{aligned}$$

$$+\sum_{0<t_i<t} e^{\beta(T-s)} G_i\big(x(t_i,x_o^k,y_o^k),y(t_i,x_o^k,y_o^k)\big)-$$
$$-\frac{\beta}{1+e^{\beta T}}\sum_{i=1}^{p} e^{\beta(T-s)} G_i\big(x(t_i,x_o^k,y_o^k),y(t_i,x_o^k,y_o^k)\big) \quad , \qquad ...\ (61)$$

Where $k=1,2$

From (60) , we have

$$\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\| = x_o^1+\int_0^t e^{\lambda(t-s)}[f\big(s,x_\infty(s,x_o^1,y_o^1),y_\infty(s,x_o^1,y_o^1)\big)-$$
$$-\int_0^T \frac{\lambda}{1+e^{\lambda T}} e^{\lambda(T-s)} f\big(s,x_\infty(s,x_o^1,y_o^1),y_\infty(s,x_o^1,y_o^1)\big)ds]ds+$$
$$+\sum_{0<t_i<t} e^{\lambda(T-s)} I_i\big(x_\infty(t_i,x_o^1,y_o^1),y_\infty(t_i,x_o^1,y_o^1)\big)-$$
$$-\frac{\lambda}{1+e^{\lambda T}}\sum_{i=1}^{p} e^{\lambda(T-s)} I_i\big(x_\infty(t_i,x_o^1,y_o^1),y_\infty(t_i,x_o^1,y_o^1)\big)-$$
$$-x_o^2-\int_0^t e^{\lambda(t-s)}[f\big(s,x_\infty(s,x_o^2,y_o^2),y_\infty(s,x_o^2,y_o^2)\big)-$$
$$-\int_0^T \frac{\lambda}{1+e^{\lambda T}} e^{\lambda(T-s)} f\big(s,x_\infty(s,x_o^2,y_o^2),y_\infty(s,x_o^2,y_o^2)\big)ds]ds-$$
$$-\sum_{0<t_i<t} e^{\lambda(T-s)} I_i\big(x_\infty(t_i,x_o^2,y_o^2),y_\infty(t_i,x_o^2,y_o^2)\big)+$$
$$+\frac{\lambda}{1+e^{\lambda T}}\sum_{i=1}^{p} e^{\lambda(T-s)} I_i\big(x_\infty(t_i,x_o^2,y_o^2),y_\infty(t,x_o^2,y_o^2)\big)\le$$
$$\le \|x_o^1-x_o^2\|+HK_1t\,\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|+$$
$$+HK_2t\,\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|+\frac{\lambda tHTK_1}{1+e^{\lambda T}}\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|+$$
$$+\frac{\lambda tHTK_2}{1+e^{\lambda T}}\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|+HK_3t\,\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|+$$
$$+HK_4t\,\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|+\frac{\lambda tHTK_3}{1+e^{\lambda T}}p\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|+$$
$$+\frac{\lambda tHTK_4}{1+e^{\lambda T}}p\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|$$
$$\le \|x_o^1-x_o^2\|+Ht\left(K_1+\frac{\lambda TK_1}{1+e^{\lambda T}}+K_3+\frac{\lambda TK_3p}{1+e^{\lambda T}}\right)\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|+$$
$$+\ Ht\left(K_2+\frac{\lambda TK_2}{1+e^{\lambda T}}+K_4+\frac{\lambda TK_4p}{1+e^{\lambda T}}\right)\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\| \quad ...\ (62)$$

By the same method and by (61), we find :

$$\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|\le\|y_o^1-y_o^2\|+$$
$$+\ Ft\left(L_1+\frac{\beta TL_1}{1+e^{\beta T}}+L_3+\frac{\beta TL_3p}{1+e^{\beta T}}\right)\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|+$$
$$+\ Ft\left(L_2+\frac{\beta TL_2}{1+e^{\beta T}}+L_4+\frac{\beta TL_4p}{1+e^{\beta T}}\right)\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\| \quad ...\ (63)$$

From (62), we have:

$$\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|\le\left(1-Ht\left(K_1+\frac{\lambda TK_1}{1+e^{\lambda T}}+K_3+\frac{\lambda TK_3p}{1+e^{\lambda T}}\right)\right)^{-1}$$
$$\left(\|x_o^1-x_o^2\|+Ht\left(K_2+\frac{\lambda TK_2}{1+e^{\lambda T}}+K_4+\frac{\lambda TK_4p}{1+e^{\lambda T}}\right)\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|\right)$$
$$...\ (64)$$

By (63), we have:

$$\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\| \le \left(1-Ft\,\left(L_2+\frac{\beta T L_2}{1+e^{\beta T}}+L_4+\frac{\beta T L_4 p}{1+e^{\beta T}}\right)\right)^{-1}$$
$$\left(\|y_o^1-y_o^2\|+Ft\,\left(L_1+\frac{\beta T L_1}{1+e^{\beta T}}+L_3+\frac{\beta T L_3 p}{1+e^{\beta T}}\right)\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|\right) \quad \ldots (65)$$

By substitute (65) in (64) , we obtain :$\overline{W_3}(1-\overline{W_4})^{-1}$

$$\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\| \le \|x_o^1-x_o^2\|+\overline{W_1}\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|+ + \overline{W_2}(1-\overline{W_4})^{-1}(\|y_o^1-y_o^2\|+\overline{W_3}\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\|)$$

So that

$$\|x_\infty(t,x_o^1,y_o^1)-x_\infty(t,x_o^2,y_o^2)\| \le (1-\overline{W_1}-\overline{W_2W_3}(1-\overline{W_4})^{-1})^{-1}(\|x_o^1-x_o^2\|+ +\overline{W_2}(1-\overline{W_4})^{-1}\|y_o^1-y_o^2\|) \quad \ldots (66)$$

And also, we have

$$\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\| \le \|y_o^1-y_o^2\|+\overline{W_4}\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|+ + \overline{W_3}(1-\overline{W_1})^{-1}(\|x_o^1-x_o^2\|+\overline{W_2}\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\|)$$

So that

$$\|y_\infty(t,x_o^1,y_o^1)-y_\infty(t,x_o^2,y_o^2)\| \le (1-\overline{W_4}-\overline{W_2W_3}(1-\overline{W_1})^{-1})^{-1}(\|y_o^1-y_o^2\|+ +\overline{W_3}(1-\overline{W_1})^{-1}\|x_o^1-x_o^2\|) \quad \ldots (67)$$

Now by substitute (66) and (67) in (58) and (59), we obtain on (56) and (57) respectively.

Remark 2 [1]:

The theorem 5 to ensure solution's to the system (1), in view of to happen small change in the point x_0 , to requite small change on the function's behavior$\Delta = \Delta(t,x_o,y_o)$.

REFERENCES

[1]. Butris R.N. Existence of periodic for non-linear systems of differential equations of operators with pulse action, Ukraine, Kiev , Math. J. No.9,pp.1260-1264,(1991).

[2]. Perestyuk N. A. Stability solutions for linear systems of pulse action, Ukraine, Kiev, Uesnki, J. , No. 19, pp 71 - 76, (1977).

[3]. Rama M. M. Ordinary differential equations theory and applications, Britain, (1981).

[4]. Ronto, N. I. Existence of periodic solution of differential equations with pulse action, Dok1. Nauk Ukraine, pp.54-55, (1987).

[5]. Samoilenko A. M. and Perestyuk N. A. Impulsive differential equations, USA: World Scientific; Ser A Nonlinear Science, vol. 14, pp.462, (1995).

[6]. Samoilenko A. M. and Ronto N. I. , A numerical – analytic methods for investigations of periodic solutions, Ukraine, Kiev, (1976).

[7]. Yan Juran, JianhuaShen, Razumikhin type stability theorems for impulsive functional differential equations, Non Linear Analysis, vol.33,pp.519-531, (1998).