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Stronger Violation of Local Realism for Three Qubit Systems with a New Bell-Type Inequality

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Abstract—In this paper we show the new Bell-type inequality for three qubit systems (i.e a tripartite system with two measurements in each side and two outputs for each measurement). This inequality is violated by quantum theory with a factor violation of 3.5 tolerating 0.5 fraction of white noise. Our inequality includes only 19 different joint probabilities whereas in other works it is much more than this. We will also show violation amount of this inequality is 2.5. Note that the maximum violation factor and amount of violation in available inequalities are both 2, also the white noise tolerance of our inequality is in agreement with maximum value calculated so far.

Keywords––non locality; Bell inequality; amount of violation; violation factor; white noise tolerance.

I. INTRODUCTION

Quantum mechanics cannot be described by local hidden variable theories. In quantum theory, the tests of local realism are based on Bell-type inequalities. Original Bell inequality did not have any capabilities to be studied empirically in the laboratories [1]. Since then, many attempts have been made to obtain Bell-type inequalities which are violated by a higher factor so that it would be experimentally easy to test the non-locality feature of quantum theory. As the non-locality feature of quantum theory is intensively used in quantum information, Bell type inequalities have received more attention in recent years [2].

A variant of Bell-type inequality, being more general and more useful for an experimental test, was derived by Clauser, Horne, Shimony and Holt (CHSH) [3]. In 1982, Aspect, et al. performed a verification experiment for a possible violation of Bell inequalities in CHSH form [4]. CHSH inequality is derived from two particles. N-particle generalizations of the CHSH inequality were proposed by Mermin, and later developed by Ardehali, Bleinski and Khyshko and others. See [5-8]. Quantum predictions can violate such inequalities by an amount increasing exponentially with the particle number [9]. for N=3, maximal violation factor (i.e. the ratio of the value of Bell-type inequalities in quantum theory to their value in local theory) and maximum amount of violation (i.e. the difference between the value of Bell expression according to quantum theory and its extermum value according to local theory) are both 2. In these systems, white noise tolerance is 0.5 [10].

In this paper, we consider a three qubit system (i.e a three particles system with two dichotomic measurements on each particle, and two outcomes for each measurement). Then for local theory, we introduce a new Bell type expression for this system based on numerical calculations. We will show that the violation factor and the amount of violation of this inequality exceed those of available inequalities [5-9], while its white noise tolerance agrees with the previous results.

II. THREE QUBIT SYSTEMS

To verify the non-locality in quantum theory We would introduce a three qubit system consisting of particles A_1, A_2 and A_3 in which two dichotomic measurements B_1^i, B_2^j and B_3^k can be performed on particles A_1, A_2 and A_3 respectively, where $i, j, k \in \{1,2\}$. The outcomes of these measurements are denoted b_1^i, b_2^j and b_3^k which can take the values 0, 1.

The local realism assumes the existence of positive triple joint probabilities involving all possible observations from which it should be possible to obtain all the quantum predictions as marginals. Let's denote these triple joint probabilities by $q_{B_1^1,B_1^2,B_2^1,B_2^2,B_3^1,B_3^2}^{b_1^1,b_1^2,b_2^1,b_2^2,b_3^1,b_3^2}$, where b_1^1 and b_1^2 represent the outcome values for measurements B_1^1 and B_1^2 on particle A_1, b_2^1 and b_2^2 represent the outcome values for measurements B_2^1 and B_2^2 on particle A_2 and b_3^1 and b_3^2 represent the outcome values for measurements B_3^1 and B_3^2 on particle A_3 respectively. These probabilities are positive. Obviously:

$$\sum_{b_1^1,b_1^2,b_2^1,b_2^2,b_3^1,b_3^2} q^{b_1^1,b_1^2,b_2^1,b_2^2,b_3^1,b_3^2}_{B_1^1,B_1^2,B_2^1,B_2^2,B_3^1,B_3^2} = 1 \qquad (1)$$

The total number of $q^{b_1^1,b_1^2,b_2^1,b_2^2,b_3^1,b_3^2}_{B_1^1,B_1^2,B_2^1,B_2^2,B_3^1,B_3^2}$'s, denoted as N_q, is 64.

Also assume $P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k}$ represents the joint probability that in a particular setting, (i.e. the probability of obtaining the values b_1^i, b_2^j and b_3^k in a simultaneous measurement of observables B_1^i, B_2^j and B_3^k on the particles A_1, A_2 and A_3 respectively).

The joint probabilities take the form:

$$P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k} = \sum_{b_1^{i'},b_2^{j'},b_3^{k'}} q^{b_1^i,b_1^{i'},b_2^j,b_2^{j'},b_3^k,b_3^{k'}}_{B_1^i,B_1^{i'},B_2^j,B_2^{j'},B_3^k,B_3^{k'}} \qquad (2)$$

where $i', j', k' \in \{1,2\}$ and $i' \neq i,\ j' \neq j$ and $k' \neq k$.

From Eqs. (1) and (2) we have $\sum_{b_1^i,b_2^j,b_3^k} P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k} = 1$. The total number of $P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k}$'s is $N_P = 64$.

As it is clear, for a local theory a Bell type expression, ß, is a linear combination of joint probabilities which can be written as:

$$ß = \sum_{I,J,K,l,m,n} \gamma^{l,m,n}_{I,J,K} P^{l,m,n}_{I,J,K} \qquad (3)$$

where $I \in \{B_1^1, B_1^2\}$, $J \in \{B_2^1, B_2^2\}$, $K \in \{B_3^1, B_3^2\}$, $l \in \{b_1^1, b_1^2\}$, $m \in \{b_2^1, b_2^2\}$ and $n \in \{b_3^1, b_3^2\}$.

Using equation (2) the Bell inequality in terms of q's would become:

$$ß = \sum_{a_1,a_2,b_1,b_2,c_1,c_2} \left(\left(\eta^{a_1,a_2,b_1,b_2,c_1,c_2} - \mu^{a_1,a_2,b_1,b_2,c_1,c_2}\right) q^{b_1^1,b_1^2,b_2^1,b_2^2,b_3^1,b_3^2}_{B_1^1,B_1^2,B_2^1,B_2^2,B_3^1,B_3^2} \right) \qquad (4)$$

It is clear that:

$$-e \leq ß \leq d \qquad (5)$$

where d(e) is the greatest of positive real numbers $\eta'(\mu's)$ in equation (5).

One of the well–known Bell type inequalities for three partite systems is Mermin inequality, which is expressed as [6]:

$$ß_M = E(B_1^1, B_2^2, B_3^2) + E(B_1^2, B_2^1, B_3^2) + E(B_1^2, B_2^2, B_3^1) - E(B_1^1, B_2^1, B_3^1) \qquad (6)$$

Where $E(B_1^i, B_2^j, B_3^k)$ is [5]:

$$E(B_1^i, B_2^j, B_3^k) = \langle B_1^i, B_2^j, B_3^k \rangle = \sum_{a_i}\sum_{b_j}\sum_{c_k} (-1)^u P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k\ k} \qquad (7)$$

In the above equation "u" is the number of zero's resulted in each particular setting.
It is shown in [6] that Mermin inequality for local theories is:

$$-2 \leq ß_M \leq 2 \qquad (8)$$

Mermin inequality satisfies in Eq. (5) for e=d=2, however, according to quantum theory, the upper bound of Mermin inequality is 4. Here, the violation factor and amount of violation are both 2 and the maximum white noise tolerance calculated is 0.5 [10, 11].

III. A NEW TREE QUBIT BELL INEQUALITY

According to local theories, various Bell expressions are defined which correspond to different values of e and d in Eq. (5). These expressions are violated by quantum theory. The violation factor and the amount of violation of original Bell

inequality, for e=1 and d=0 in Eq. (5), are 1.414 and 0.414 respectively. In [12] several Bell type inequalities, with different values of e and d, are introduced. The numerical results in [12, 13] show that amount of violation and the violation factor of inequalities, as defined in section 2, are related to e and d, and maximum violation increases when the upper/lower bound in Eq. (5) increases/decreases.
One of these Bell expressions, for a tow qubit system, is as follows [12]:

$$-1 \leq \begin{pmatrix} -P^{0,0}_{B^1_1,B^1_2} + P^{0,1}_{B^1_1,B^1_2} + P^{1,0}_{B^1_1,B^1_2} + P^{0,0}_{B^1_1,B^2_2} - P^{1,0}_{B^1_1,B^2_2} \\ + P^{1,1}_{B^1_1,B^2_2} + P^{0,1}_{B^2_1,B^1_2} - 2P^{1,1}_{B^2_1,B^1_2} - P^{0,0}_{B^2_1,B^2_2} + 2P^{1,0}_{B^2_1,B^2_2} \end{pmatrix} \leq 2 \tag{9}$$

where B^i_1 and B^j_2 are two possible measurements performed on particles A_1 and A_2 respectively and $i, j \in \{1,2\}$. With this notation, outcomes of these measurements are b^i_1 and b^j_2 which $b^i_1, b^j_2 \in \{0,1\}$. So $P^{b^i_1,b^j_2}_{B^i_1,B^j_2}$ denotes the probability that in a particular experiment, measurement B^i_1 on particle A_1 results b^i_1 and measurement B^i_2 on particle A_2 results b^j_2. The calculated violation factor and amount of violation, for inequality (9), are 1.621 and 0.621 respectively which are more than the previous results in the literature.
Inspired by the result obtained in [12, 13], we have looked for three qubit Bell- type inequality in local theories which violates quantum theory with stronger violation.
Writing Eq. (2) in matrix form, i.e. $\mathbf{P} = \mathbf{Uq}$, where $\mathbf{P}$ is an $N_P \times 1$ matrix, N_P is the total number of joint probabilities, $\mathbf{q}$ is an $N_q \times 1$ matrix, the total number of triple joint probabilities is N_q, and **U** is the conversion matrix with dimension $N_P \times N_q$. Using numerical method mentioned in [12, 13] we have solved equation $\mathbf{P} = \mathbf{Uq}$ to find all possible Bell expressions that satisfy eq. (5). Using the three qubit GHZ state, we obtain the values of derived Bell expressions in quantum theory. The violation factor and the amount of violation for the expressions that are violated by quantum theory have been calculated and the inequality that is violated more strongly than those other expressions in quantum theory has been obtained. However we don't discuss these here, because the result that we are going to use in following can be tested directly and easily.
Let's consider $B^1_\mu = x$ and $B^2_\mu = y$ where $\mu = 1,2,3$, we find the following Bell expression with the help of numerical calculations mentioned above:

$$\begin{aligned} ß_{new} = &+P^{001}_{yyy} + P^{110}_{yyy} + 4P^{011}_{yyy} + 4P^{100}_{yyy} - 4P^{111}_{xyy} - 4P^{100}_{xyy} + P^{000}_{xyy} + P^{011}_{xyy} - 5P^{100}_{yxy} - 5P^{001}_{yxy} \\ &- 4P^{100}_{yyx} - 4P^{010}_{yyx} - P^{001}_{yyx} - P^{111}_{yyx} + 5P^{100}_{xxx} + P^{111}_{xxx} + P^{001}_{xxx} + P^{101}_{xxx} - 4P^{011}_{xxx} \end{aligned} \tag{10}$$

In appendix A, it is shown that:

$$ß_{new} \leq 1 \tag{11}$$

Now we show that inequality (11) is violated by quantum theory. Consider a three-qubit Greenberger-Horne-Zeilinger state [14] which is:

$$|\Psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle) \tag{12}$$

where $\uparrow$ and $\downarrow$ are spin polarization along z axis. So in quantum theory, $ß_{new}$ would become:

$$ß_{new} = \frac{1}{8} + \frac{1}{8} + \frac{4}{8} + \frac{4}{8} - 0 - 0 + \frac{1}{4} + \frac{1}{4} - 0 - 0 - 0 - 0 - 0 - 0 + \frac{5}{4} + \frac{1}{4} + \frac{1}{4} + 0 - 0 = \frac{7}{2} \tag{13}$$

It can be seen that the violation factor and the amount of violation of the inequality (11) are 3.5 and 2.5 respectively, whereas the maximum violation factor and maximum amount of violation of the available inequalities so far, are both 2.
To calculate the white noise tolerance of $ß_{new}$ in three qubit systems, we consider the following density matrix [11]:

$$\rho_{WhiteNoise} = (1-\nu)|\Psi\rangle_{GHZ\ GHZ}\langle\Psi| + \frac{\nu}{8} I \tag{14}$$

where ν is the tolerance of the Bell expression, i.e. the maximum fraction of white noise admixture for which a Bell expression stops being violated.
Obviously:

$$P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k} = (1-\nu)\, P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k} + \frac{\nu}{8} \qquad (15)$$

$P^{b_1^i,b_2^j,b_3^k}_{B_1^i,B_2^j,B_3^k}$ is the joint probability in the presence of white noise. From equations (11) and (15), we have:

$$\nu = \frac{ß^{QM}_{new} - ß^{L}_{new}}{ß^{QM}_{new} - \frac{f-g}{8}} \qquad (16)$$

where $ß^{QM}_{new}$ is the value of our Bell expression, $ß_{new}$, according to quantum theory and $ß^{L}_{new}$ is the upper bound of $ß_{new}$ according to local theory, and f(g) is the number of positive(negative) terms in Bell expression.

So the white noise tolerance of our equality, i.e. Eq. (11), is:

$$\nu = \frac{\frac{7}{2} - 1}{\frac{7}{2} - \frac{20-32}{8}} = 0.5 \qquad (17)$$

which agrees with maximum value calculated up to now.

IV. CONCLUSION

In this paper, it is shown that a new bell-type inequality exists for the three-qubit system, i.e. a three particles system with two measurements for each side and two outputs for each measurement, in local theories and shown that this inequality is violated by quantum theory with a violation factor of 3.5 and violation amount of 2.5. We have also shown that white noise tolerance in this new Bell type expression is 0.5. Two typical bipartite qubit inequalities are CHSH and CH inequalities in which the violation factor is 0.414 and the tolerance of white noise is 0.292. The bipartite qutrits inequalities have been studied widely in recent years. Their maximum value of the amount of violation and the tolerance are predicted to be 0.87293 and 0.30385 respectively for maximally entangled state. Note that for inequalities in three qubit systems, the violation factor and amount of violation predicted are both 2 which are less than those of our inequality. Also the maximum white noise tolerance of available three qubit inequalities is 0.5, which agrees with white noise tolerance of our inequality. As no experiment is error- free, there was an endeavor to gain a kind of Bell type inequality that could be violated as much as possible, and would be experimentally easy to test non-locality feature of quantum theory, so in our inequality, this increment of violation factor and the amount of violation increase the accuracy of experiments in which the errors are inevitable. Also one of the advantages of our inequality is that it includes only 19 different terms of joint probabilities whereas in other works it is much more than this (for example in Mermin and Svetlichny inequalities, it is 32 and 64 respectively). So, our inequality requires less measurement which in turn, reduces the errors due to experiment. See [13].

Appendix A

In this appendix we derive equation (11) From the definition (2) and denoting

$q^{b_1^1,b_1^2,b_2^1,b_2^2,b_3^1,b_3^2}_{B_1^1,B_1^2,B_2^1,B_2^2,B_3^1,B_3^2} = q^{b_1^1,b_1^2,b_2^1,b_2^2,b_3^1,b_3^2}$, for simplicity, we have:

$$P^{110}_{yyy} = q^{010100} + q^{010110} + q^{011100} + q^{011110} + q^{110100} + q^{110110} + q^{111100} + q^{111110}$$

$$P^{001}_{yyy} = q^{000001} + q^{000011} + q^{001001} + q^{001011} + q^{100001} + q^{100011} + q^{101001} + q^{101011}$$

$$P^{100}_{yyy} = q^{010000} + q^{010010} + q^{011000} + q^{011010} + q^{110000} + q^{110010} + q^{111000} + q^{111010}$$

$$P^{011}_{yyy} = q^{000101} + q^{000111} + q^{001101} + q^{001111} + q^{100101} + q^{100111} + q^{101101} + q^{101111}$$

$$p^{000}_{xyy} = q^{000000} + q^{000010} + q^{001000} + q^{001010} + q^{010000} + q^{010010} + q^{011000} + q^{011010}$$

$$P^{011}_{xyy} = q^{000101} + q^{000111} + q^{001101} + q^{001111} + q^{010101} + q^{010111} + q^{011101} + q^{011111}$$

$$P^{111}_{xyy} = q^{010110} + q^{010111} + q^{011110} + q^{011111} + q^{110110} + q^{110111} + q^{111110} + q^{111111}$$

$$p^{100}_{xyy} = q^{100000} + q^{100010} + q^{101000} + q^{101010} + q^{110000} + q^{110010} + q^{111000} + q^{111010}$$

$$P^{001}_{yyx} = q^{000010} + q^{000011} + q^{001010} + q^{001011} + q^{100010} + q^{100011} + q^{101010} + q^{101011}$$

$$P^{010}_{yyx} = q^{000100} + q^{000101} + q^{001100} + q^{001101} + q^{100100} + q^{100101} + q^{101100} + q^{101101}$$

$$P_{yyx}^{100} = q^{010000} + q^{010001} + q^{011000} + q^{011001} + q^{110000} + q^{110001} + q^{111000} + q^{111001}$$

$$P_{yyx}^{111} = q^{010110} + q^{010111} + q^{011110} + q^{011111} + q^{110110} + q^{110111} + q^{111110} + q^{111111}$$

$$P_{XXX}^{1,1,1} = q^{101010} + q^{101011} + q^{101110} + q^{101111} + q^{111010} + q^{111011} + q^{111110} + q^{111111}$$

$$P_{XXX}^{1,0,0} = q^{100000} + q^{100001} + q^{100100} + q^{100101} + q^{110000} + q^{110001} + q^{110100} + q^{110101}$$

$$P_{XXX}^{0,0,1} = q^{000010} + q^{000011} + q^{000110} + q^{000111} + q^{010010} + q^{010011} + q^{010110} + q^{010111}$$

$$P_{XXX}^{1,0,1} = q^{100010} + q^{100011} + q^{100110} + q^{100111} + q^{110010} + q^{110011} + q^{110110} + q^{110111}$$

$$P_{XXX}^{0,1,1} = q^{001010} + q^{001011} + q^{001110} + q^{001111} + q^{011010} + q^{011011} + q^{011110} + q^{011111}$$

$$P_{yxy}^{100} = q^{010000} + q^{010010} + q^{010100} + q^{010110} + q^{110000} + q^{110010} + q^{110100} + q^{110110}$$

$$P_{yxy}^{001} = q^{000001} + q^{000011} + q^{000101} + q^{000111} + q^{100001} + q^{100011} + q^{100101} + q^{100111}$$

After simplifying we obtain:

$$\begin{aligned} ß_{new} = & -4q^{111111} - 4q^{011111} + q^{101111} - 4q^{110111} + q^{111011} - 4q^{111101} + q^{111110} + q^{001111} \\ & + q^{010111} - 4q^{011011} + q^{011101} - 4q^{011110} - 4q^{100111} + q^{101011} - 4q^{101101} + q^{101110} \\ & + q^{110011} + q^{110101} - 4q^{110110} - 4q^{111001} + q^{111010} + q^{111100} - 4q^{101100} - 4q^{001011} \\ & - 4q^{100011} - 4q^{100101} + q^{100110} - 4q^{101010} + q^{010101} + q^{010011} - 4q^{010110} + q^{101001} \\ & + q^{000000} + q^{100000} - 4q^{010000} + q^{001000} - 4q^{000100} + q^{000010} - 4q^{000001} - 4q^{110000} \\ & - 4q^{101000} + q^{100100} - 4q^{100010} + q^{100001} + q^{011000} - 4q^{010100} + q^{010010} - 4q^{010001} \\ & - 4q^{001100} - 4q^{001010} + q^{001001} + q^{000110} - 4q^{000101} - 4q^{000011} - 4q^{111000} + q^{110100} \\ & - 4q^{110010} + q^{110001} + q^{011100} + q^{011010} - 4q^{011001} - 4q^{001110} + q^{0011010} + q^{000111} \end{aligned}$$

Please note that all q's are positive here, so according to equation 12, $ß_{new}$ is less than or equal to 1.

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