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The Finite Difference Methods And Its Stability For Glycolysis Model In Two Dimensions

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Abstract: *The Glycolysis Model Has Been Solved Numerically In Two Dimensions By Using Two Finite Differences Methods: Alternating Direction Explicit And Alternating Direction Implicit Methods (ADE And ADI) And We Were Found That The ADE Method Is Simpler While The ADI Method Is More Accurate. Also, We Found That ADE Method Is Conditionally Stable While ADI Method Is Unconditionally Stable.*
Keywords: *Glycolysis Model, ADE Method, ADI Method.*

I. INTRODUCTION

Chemical reactions are modeled by non-linear partial differential equations (PDEs) exhibiting travelling wave solutions. These oscillations occur due to feedback in the system either chemical feedback (such as autocatalysis) or temperature feedback due to a non-isothermal reaction.

Reaction-diffusion (RD) systems arise frequently in the study of chemical and biological phenomena and are naturally modeled by parabolic partial differential equations (PDEs). The dynamics of RD systems has been the subject of intense research activity over the past decades. The reason is that RD system exhibit very rich dynamic behavior including periodic and quasi-periodic solutions and chaos(see, for example [8].

1.1. MATHEMATICAL MODEL:

A general class of nonlinear-diffusion system is in the form

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= d_1\Delta u + a_1 u + b_1 v + f(u,v) + g_1(x)\\ \frac{\partial v}{\partial t} &= d_2\Delta u + a_2 u + b_2 v - f(u,v) + g_2(x)\end{aligned}\right\} \quad (1)$$

with homogenous Dirchlet or Neumann boundary condition on a bounded domain Ω , n≤3, with locally Lipschitz continuous boundary. It is well known that reaction and diffusion of chemical or biochemical species can produce a variety of spatial patterns. This class of reaction diffusion systems includes some significant pattern formation equations arising from the modeling of kinetics of chemical or biochemical reactions and from the biological pattern formation theory.

In this group, the following four systems are typically important and serve as mathematical models in physical chemistry and in biology:

- **Brusselator model:**
  $$a_1 = -(b+1), b_1 = 0, a_2 = b, b_2 = 0, f = u^2 v, g_1 = a, g_2 = 0,$$
  where *a* and *b* are positive constants.
- **Gray-Scott model:**
  $$a_1 = -(f+k), b_1 = 0, a_2 = 0, b_2 = -F, f = u^2 v, g_1 = 0, g_2 = F,$$
  where *F* and *k* are positive constants.
- **Glycolysis model:**
  $$a_1 = -1, b_1 = k, a_2 = 0, b_2 = -k, f = u^2 v, g_1 = \rho, g_2 = \delta,$$
- where $k, p,$ and δ are positive constants
- **Schnackenberg model:**
  $$a_1 = -k, b_1 = a_2 = b_2 = 0, f = u^2 v, g_1 = a, g_2 = b,$$

where *k*, *a* and *b* are positive constants.

Then one obtains the following system of two nonlinearly coupled reaction-diffusion equations (the Glycolysis model),

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= d_1\Delta u - u + kv + u^2 v + \rho, \quad (t,x) \in (0,\infty)\times\Omega\\ \frac{\partial v}{\partial t} &= d_2\Delta v - kv - u^2 v + \delta, (t,x) \in (0,\infty)\times\Omega\\ u(t,x) &= v(t,x) = 0, \quad t>0, \quad x\epsilon\partial\Omega\\ u(0,x) &= u_0(x), \quad v(0,x) = v_0(x), \quad x\epsilon\Omega\end{aligned}\right\} \quad (2)$$

where $d_1, d_2, p, k\ and\ \delta$ are positive constants [9].

II. MATERIALS AND METHODS

2.1. Derivation Of Alternating Direction Explicit (ADE) For Glycolysis Model:

The two dimensional Glycolysis model is given by

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= d_1\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-u+Kv+u^2v+\rho, \qquad (t,x)\in(0,\infty)\times\Omega\\ \frac{\partial v}{\partial t} &= d_2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-Kv-u^2v+del, \qquad (t,x)\in(0,\infty)\times\Omega\end{aligned}\right\} \quad (2)$$

We consider a square region $0\le x\le 1$, $0\le y\le 1$ and u, v are known at all points within and on the boundary of the square region. We draw lines parallel to x, y, t–axis as $x=ph$, $y=qk$, and $t=nz$, $p,q=0, 1, 2,\ldots,M$ and $n=0, 1, 2,\ldots, N$, where $h=\delta x$, $k=\delta y$, $z=\delta t$.

Denote the values of u at these mesh points by $u(ph,qk,nz)=u_{p,q,n}$ and $v(ph,qk,nz)=v_{p,q,n}$. The explicit finite difference representation of the Glycolysis model is

$$\frac{u_{p,q,n+1}-u_{p,q,n}}{z}=\frac{d_1}{h^2}(u_{p+1,q,n}-2u_{p,q,n}+u_{p-1,q,n}+u_{p,q+1,n}-2u_{p,q,n}+u_{p,q-1,n})-u_{p,q,n}+Kv_{p,q,n}+u^2_{p,q,n}v_{p,q,n}+\rho$$

$$\frac{v_{p,q,n+1}-v_{p,q,n}}{z}=\frac{d_2}{h^2}(v_{p+1,q,n}-2v_{p,q,n}+v_{p-1,q,n}+v_{p,q+1,n}-2v_{p,q,n}+v_{p,q-1,n})-Kv_{p,q,n}-u^2_{p,q,n}v_{p,q,n}+del,$$

and

$$u_{p,q,n+1}-u_{p,q,n}=\frac{d_1 z}{h^2}(u_{p+1,q,n}-2u_{p,q,n}+u_{p-1,q,n}+u_{p,q+1,n}-2u_{p,q,n}+u_{p,q-1,n})-zu_{p,q,n}+zKv_{p,q,n}+zu^2_{p,q,n}v_{p,q,n}+\rho z$$

$$v_{p,q,n+1}-v_{p,q,n}=\frac{d_2 z}{h^2}(v_{p+1,q,n}-2v_{p,q,n}+v_{p-1,q,n}+v_{p,q+1,n}-2v_{p,q,n}+v_{p,q-1,n})-zKv_{p,q,n}-zu^2_{p,q,n}v_{p,q,n}.+del$$

Assume that $h=k$ and $m_i=\frac{d_i z}{h^2}, i=1, 2.$ Then simplifying the system to obtain

$$u_{p,q,n+1}-u_{p,q,n}=m_1(u_{p+1,q,n}-2u_{p,q,n}+u_{p-1,q,n}+u_{p,q+1,n}-2u_{p,q,n}+u_{p,q-1,n})-zu_{p,q,n}+zKv_{p,q,n}+zu^2_{p,q,n}v_{p,q,n}+\rho z$$

$$v_{p,q,n+1}-v_{p,q,n}=m_2(v_{p+1,q,n}-2v_{p,q,n}+v_{p-1,q,n}+v_{p,q+1,n}-2v_{p,q,n}+v_{p,q-1,n})-zKv_{p,q,n}-zu^2_{p,q,n}v_{p,q,n}+del,$$

and

$$u_{p,q,n+1}=(1-4m_1-z(B+1))u_{p,q,n}+m_1(u_{p+1,q,n}+u_{p-1,q,n}+u_{p,q+1,n}+u_{p,q-1,n})+zu_{p,q,n}z+zKv_{p,q,n}+zu^2_{p,q,n}v_{p,q,n}+\rho z$$

$$v_{p,q,n+1}=(1-4m_2)v_{p,q,n}+(v_{p+1,q,n}+u_{p-1,q,n}+u_{p,q+1,n}+u_{p,q-1,n})m_2-zKv_{p,q,n}-zu^2_{p,q,n}v_{p,q,n}.+del$$

This is the alternating direction explicit formula for the Glycolysis model

Derivation Of Alternating Direction Implicit (ADI)For Glycolysis Model:

In the ADI approach, the finite difference equations are written in terms of quantities at two x levels. However, two different finite difference approximations are used alternately, one to advance the calculations from the plane n to a plane $n+1$, and the second to advance the calculations from $(n+1)$-plane to the $(n+2)$-plane by replacing $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 v}{\partial x^2}$ by implicit finite difference approximation [5]. we get

$$\frac{u_{p,q,n+1}-u_{p,q,n}}{z}=\frac{d_1}{h^2}(u_{p+1,q,n+1}-2u_{p,q,n+1}+u_{p-1,q,n+1})+\frac{d_1}{k^2}(u_{p,q+1,n}-2u_{p,q,n}+u_{p,q-1,n})-u_{p,q,n}+Kv_{p,q,n}+ u^2_{p,q,n}v_{p,q,n}+\rho,$$

$$\frac{v_{p,q,n+1}-v_{p,q,n}}{z}=\frac{d_2}{h^2}(v_{p+1,q,n+1}-2v_{p,q,n+1}+v_{p-1,q,n+1})+\frac{d_2}{K^2}(v_{p,q+1,n}-2v_{p,q,n}+v_{p,q-1,n})-Kv_{p,q,n}-u^2_{p,q,n}v_{p,q,n}+del,$$

and

$$\frac{u_{p,q,n+2}-u_{p,q,n+1}}{z}=\frac{d_1}{h^2}(u_{p+1,q,n+1}-2u_{p,q,n+1}+u_{p-1,q,n+1})+\frac{d_1}{k^2}(u_{p,q+1,n+2}-2u_{p,q,n+2}+u_{p,q-1,n+2})-u_{p,q,n}+Kv_{p,q,n}+ u^2_{p,q,n}v_{p,q,n}+\rho,$$

$$\frac{v_{p,q,n+2}-v_{p,q,n+1}}{z}=\frac{d_2}{h^2}(v_{p+1,q,n+1}-2v_{p,q,n+1}+v_{p-1,q,n+1})+\frac{d_2}{K^2}(v_{p,q+1,n+2}-2v_{p,q,n+2}+v_{p,q-1,n+2})-Kv_{p,q,n}-u^2_{p,q,n}v_{p,q,n}+del.$$

Simplifying the system will give

$$u_{p,q,n+1} = \frac{d_1 z}{h^2}(u_{p+1,q,n+1} - 2u_{p,q,n+1} + u_{p-1,q,n+1}) + \frac{d_1 z}{k^2}(u_{p,q+1,n} - 2u_{p,q,n} + u_{p,q-1,n}) - zu_{p,q,n} + zu^2_{p,q,n}v_{p,q,n} + zKv_{p,q,n}$$
$$\rho z + u_{p,q,n}$$

$$v_{p,q,n+1} = \frac{d_2 z}{h^2}(v_{p+1,q,n+1} - 2v_{p,q,n+1} + v_{p-1,q,n+1}) + \frac{d_2 z}{K^2}(v_{p,q+1,n} - 2v_{p,q,n} + v_{p,q-1,n}) - Kv_{p,q,n} - zu^2_{p,q,n}v_{p,q,n} + zv_{p,q.n+1} + zdel.$$

From the second two equations we will obtain a system of the form

$$u_{p,q,n+2} = \frac{d_1 z}{h^2}(u_{p+1,q,n+1} - 2u_{p,q,n+1} + u_{p-1,q,n+1}) + \frac{d_1 z}{k^2}(u_{p,q+1,n}+2 - 2u_{p,q,n}+2 + u_{p,q-1,n}+2) - zu_{p,q,n} + zu^2_{p,q,n}v_{p,q,n} + zKv_{p,q,n}$$
$$\rho z + u_{p,q,n}$$

$$v_{p,q,n+2} = \frac{d_2 z}{h^2}(v_{p+1,q,n+1} - 2v_{p,q,n+1} + v_{p-1,q,n+1}) + \frac{d_2 z}{k^2}(v_{p,q+1,n+2} - 2v_{p,q,n+2} + v_{p,q-1,n+2}) - zKv_{p,q,n} - zu^2_{p,q,n}v_{p,q,n} + zv_{p,q,n+1} + zdel.$$

From the first two equations where $r_1 = \frac{d_1 z}{h^2}$, $r_2 = \frac{d_1 z}{k^2}$, $m_1 = \frac{d_2 z}{h^2}$ and $m_2 = \frac{d_2 z}{k^2}$, we get

$$u_{p,q,n+1} = r_1(u_{p+1,q,n+1} - 2u_{p,q,n+1} + u_{p-1,q,n+1}) + r_2(u_{p,q+1,n} - 2u_{p,q,n} + u_{p,q-1,n}) - zu_{p,q,n} + zKv_{p,q,n}$$
$$zu^2_{p,q,n}v_{p,q,n} + \rho z + u_{p,q,n},$$

$$v_{p,q,n+1} = m_1(v_{p+1,q,n+1} - 2v_{p,q,n+1} + v_{p-1,q,n+1}) + m_2(v_{p,q+1,n} - 2v_{p,q,n} + v_{p,q-1,n}) - zKv_{p,q,n} -$$
$$zu^2_{p,q,n}v_{p,q,n} + zdel + v_{p,q.n},$$

and this implies that

$$u_{p,q,n+1} = r_1(u_{p+1,q,n+1} - 2u_{p,q,n+1} + u_{p-1,q,n+1}) + r_2(u_{p,q+1,n} + u_{p,q-1,n}) + (1 - 2r_1 - z)u_{p,q,n} + zu^2_{p,q,n}v_{p,q,n} + \rho z + zKv_{p,q,n}$$

$$v_{p,q,n+1} = m_1(v_{p+1,q,n+1} - 2v_{p,q,n+1} + v_{p-1,q,n+1}) + m_2(v_{p,q+1,n} + v_{p,q-1,n}) + (1 - 2m_2)v_{p,q,n} - zu^2_{p,q,n}v_{p,q,n} \cdot + zdel$$

We have simplifying these systems of equations yields

$$(1 + 2r_1)u_{p,q,n+1} = r_1(u_{p+1,q,n+1} + u_{p-1,q,n+1}) + r_2(u_{p,q+1,n} + u_{p,q-1,n}) + (1 - 2r_2 - z)u_{p,q,n} +$$
$$\rho z + zKv_{p,q,n} + zu^2_{p,q,n}v_{p,q,n}$$

$$(1 + 2m_1)v_{p,q,n+1} = m_1(v_{p+1,q,n+1} + v_{p-1,q,n+1}) + m_2(v_{p,q+1,n} + v_{p,q-1,n}) + (1 - 2m_2)v_{p,q,n} -$$
$$zu^2_{p,q,n}v_{p,q,n},$$

also the second system of equations, yields

$$(1 + 2r_2)u_{p,q,n+2} = r_1(u_{p+1,q,n+1} + u_{p-1,q,n+1}) + (1 - 2r_1)u_{p,q,n+1} + r_2(u_{p,q+1,n+2} + u_{p,q-1,n+2}) - zu_{p,q,n} +$$
$$zu^2_{p,q,n}v_{p,q,n} + zKv_{p,q,n} + \rho z$$

$$(1 + 2m_2)v_{p,q,n+2} = m_1(v_{p+1,q,n+1} + v_{p-1,q,n+1}) + (1 - 2m_1)v_{p,q,n+1} + m_2(v_{p,q+1,n+2} + v_{p,q-1,n+2}) -$$
$$zu^2_{p,q,n}v_{p,q,n} + zdel.$$

The last two systems represent Alternating Direction implicit under the conditions

$$u_{1,q,n+1} = u_{1,q+1,n+1} = 0$$
$$u_{M,q,n+1} = u_{M,q+1,n+1} = 0.$$

Also we have

$$v_{1,q,n+1} = v_{1,q+1,n+1} = 0$$
$$v_{M,q,n+1} = v_{M,q+1,n+1} = 0.$$

The tridiagonal matrices for the system in the level *n* advanced to the level *n*+1, for both *u* and *v* can be formulated as follows *AU=B*.

$$\begin{bmatrix} (1+2r_1) & -r_1 & 0 & 0 & 0 & . & . & . & 0 \\ -r_1 & (1+2r_1) & -r_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r_1 & (1+2r_1) & -r_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_1 & (1+2r_1) & -r_1 & & & & \\ . & 0 & 0 & -r_1 & . & . & & & \\ . & . & 0 & 0 & . & . & . & & \\ . & . & . & 0 & 0 & . & . & . & \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & -r_1 & (1+2r_1) & -r_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r_1 & (1+2r_1) \end{bmatrix} \begin{bmatrix} u_{2,q,n+1} \\ u_{3,q,n+1} \\ u_{4,q,n+1} \\ u_{5,q+1} \\ . \\ . \\ . \\ \\ u_{n-3,q+1} \\ u_{n-2,q+1} \\ u_{M-1,q,n+1} \end{bmatrix} =$$

$$\begin{bmatrix} \rho z + r_2(u_{2,q+1,n} + u_{2,q-1,n}) + (1-2r_2 - z)u_{2,q,n} + zu^2_{2,q,n}v_{2,q,n} \\ \rho z + r_2(u_{3,q+1,n} + u_{3,q-1,n}) + (1-2r_2 - z)u_{3,q,n} + zu^2_{3,q,n}v_{3,q,n} \\ \rho z + r_2(u_{4,q+1,n} + u_{4,q-1,n}) + (1-2r_2 - z)u_{4,q,n} + zu^2_{4,q,n}v_{4,q,n} \\ \\ . \\ . \\ . \\ . \\ \\ \rho z + r_2(u_{M-1,q+1,n} + u_{M-1,q-1,n}) + (1-2r_2 - z)u_{M-1,q,n} + zu^2_{M-1,q,n}v_{M-1,q,n} \end{bmatrix}$$

$$\begin{bmatrix} (1+2m_1) & -m_1 & 0 & 0 & 0 & . & . & . & 0 \\ -m_1 & (1+2m_1) & -m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -m_1 & (1+2m_1) & -m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -m_1 & (1+2m_1) & -m_1 & & & & \\ . & 0 & 0 & -m_1 & . & . & & & \\ . & . & 0 & 0 & . & . & . & & \\ . & . & . & 0 & 0 & . & . & . & \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & -m_1 & (1+2m_1) & -m_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m_1 & (1+2m_1) \end{bmatrix} \begin{bmatrix} v_{2,q,n+1} \\ v_{3,q,n+1} \\ v_{4,q,n+1} \\ v_{5,q,n+1} \\ . \\ . \\ . \\ \\ v_{M-3,q,n+1} \\ v_{M-2,q,n+1} \\ v_{M-1,q,n+1} \end{bmatrix} =$$

$$\begin{bmatrix} m_2(v_{2,q+1,n} + v_{2,q-1,n}) + (1-2m_2 - zK)v_{2,q,n} - zu^2_{2,q,n}v_{2,q,n} \\ m_2(v_{3,q+1,n} + v_{3,q-1,n}) + (1-2m_2 - zK)v_{3,q,n} - zu^2_{3,q,n}v_{3,q,n} \\ m_2(v_{4,q+1,n} + v_{4,q-1,n}) + (1-2m_2 - zK)v_{4,q,n} - zu^2_{4,q,n}v_{4,q,n} \\ \\ . \\ . \\ . \\ . \\ \\ m_2(v_{M-1,q+1,n} + v_{M-1,q-1,n}) + (1-2m_2 - zK)v_{M-1,q,n} - zu^2_{M-1,q,n}v_{M-1,q,n} \end{bmatrix}$$

And the tridiagonal matrices for the system in level n+1 advanced to level n+2 for both u and v are given by AU=B.

$$\begin{bmatrix} (1+2r_2) & -r_2 & 0 & 0 & 0 & . & . & . & 0 \\ -r_2 & (1+2r_2) & -r_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r_1 & (1+2r_1) & -r_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_2 & (1+2r_2) & -r_2 & & & & \\ . & 0 & 0 & -r_2 & . & . & & & \\ . & . & 0 & 0 & . & . & . & & \\ . & . & . & 0 & 0 & . & . & . & \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & -r_2 & (1+2r_2) & -r_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r_2 & (1+2r_2) \end{bmatrix} \begin{bmatrix} u_{p,2,n+2} \\ u_{p,3,n+2} \\ u_{p,4,n+2} \\ u_{p,5,n+2} \\ . \\ . \\ . \\ \\ u_{p,M-3,n+2} \\ u_{p,M-2,n+2} \\ u_{p,M-1,n+2} \end{bmatrix} =$$

$$\begin{bmatrix} \rho z + r_1(u_{p+1,2,n+1} + u_{p-1,2,n+1}) + (1-2r_1 - z)u_{p,2,n+1} - zu_{p,2,n} + zu^2_{p,2,n}v_{p,2,n} \\ \rho z + r_1(u_{p+1,3,n+1} + u_{p-1,3,n+1}) + (1-2r_1 - z)u_{p,3,n+1} - zu_{p,3,n} + zu^2_{p,3,n}v_{p,3,n} \\ \rho z + r_1(u_{p+1,4,n+1} + u_{p-1,4,n+1}) + (1-2r_1 - z)u_{p,4,n+1} - zu_{p,4,n} + zu^2_{p,4,n}v_{p,4,n} \\ \rho z + r_1(u_{p+1,5,n+1} + u_{p-1,5,n+1}) + (1-2r_1 - z)u_{p,5,n+1} - zu_{p,5,n} + zu^2_{p,5,n}v_{p,5,n} \\ . \\ . \\ . \\ . \\ \rho z + r_1(u_{p+1,M-1,n+1} + u_{p-1,M-1,n+1}) + (1-2r_1 - z)u_{p,M-1,n+1} - zu_{pM-1,2,n} + zu^2_{p,2,n}v_{pM-1,n+1} \end{bmatrix}$$

Also for AV=B the tridiagonal is in the form

$$\begin{bmatrix} (1+2m_2) & -m_2 & 0 & 0 & 0 & . & . & . & 0 \\ -m_2 & (1+2m_2) & -m_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -m_2 & (1+2m_2) & -m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -m_2 & (1+2m_2) & -m_2 & & & & \\ . & 0 & 0 & -m_1 & . & . & & & \\ . & . & 0 & 0 & . & . & . & & \\ . & . & . & 0 & 0 & . & . & . & \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & -m_2 & (1+2m_2) & -m_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m_2 & (1+2m_2) \end{bmatrix} \begin{bmatrix} v_{p,2,n+2} \\ v_{p,3,n+2} \\ v_{p,4,n+2} \\ v_{p,5,n+2} \\ . \\ . \\ . \\ v_{p,M-3,n+2} \\ v_{p,M-2,n+2} \\ v_{p,M-1,n+2} \end{bmatrix} =$$

$$\begin{bmatrix} m_1(v_{p+1,2,n+1} + v_{p-1,2,n+1}) + (1-2m_1 - zK)v_{p,2,n+1} + zu_{p,2,n} - zu^2_{p,2,n}v_{p,2,n} \\ m_1(v_{p+1,3,n+1} + v_{p-1,3,n+1}) + (1-2m_1 - zK)v_{p,3,n+1} + zu_{p,3,n} - zu^2_{p,3,n}v_{p,3,n} \\ m_1(v_{p+1,4,n+1} + v_{p-1,4,n+1}) + (1-2m_1 - zK)v_{p,4,n+1} + zu_{p,4,n} - zu^2_{p,4,n}v_{p,4,n} \\ . \\ . \\ . \\ . \\ m_1(v_{p+1,M-1,n+1} + v_{p-1,M-1,n+1}) + (1-2m_1 - zK)v_{p,M-1,n+1} + zu_{p,M-1,n} - zu^2_{p,M-1,n}v_{p,M-1,n} \end{bmatrix}$$

2.2. Numerical Stability In Two Dimensional Spaces:

2.3.1 Numerical stability of ADE: The Von-Neumann method has been used to study the stability analysis of Glycolysis model in two dimensions; we can apply this method by substituting the solution in finite difference method at time t by $\psi(t)e^{m\beta x}e^{m\gamma y}$, where $\beta,\ \gamma > 0$ and $m = \sqrt{-1}$. To apply Von-Neumann on the first equation of Glycolysis model

$$\frac{\partial u}{\partial t} = d_1[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}] - u + Kv + u^2 v + \rho,$$

$$\frac{\partial v}{\partial t} = d_2[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}] - Kv - u^2 v + del,$$

For the first equation after linearizing it and for some values of ρ and K neglect the terms ρz and $zKv_{p,q,n}$ [3], will be in the form

$$\psi(t+\Delta t)e^{m\beta x}e^{m\gamma y} = (1-4r_1-z)\psi(t)e^{m\beta x}e^{m\gamma y} + r_1(\psi(t)e^{m\beta(x+\Delta x)}e^{m\gamma y} +$$
$$\psi(t)e^{m\beta(x-\Delta x)}e^{m\gamma y} + \psi(t)e^{m\beta x}e^{m\gamma(y+\Delta y)} + \psi(t)e^{m\beta x}e^{m\gamma(y-\Delta y)}.$$

Now dividing both sides of the above equation by $\psi(t)e^{m\beta x}e^{m\gamma y}$ to obtain

$$\frac{\psi(t+\Delta t)}{\psi(t)} = (1-4r_1-z) + r_1(e^{m\beta\Delta x} + e^{-m\beta\Delta x} + e^{m\beta\Delta y} + e^{-m\beta\Delta y})$$
$$= (1-4r_1-z) + r_1(2\cos(\beta\Delta x) + 2\cos(\gamma\Delta y)$$
$$= (1-4r_1-z) + r_1(1-2\sin^2(\beta\Delta x/2) + 1 - 2\sin^2(\gamma\Delta y/2)).$$

For some values of β and γ, $\sin^2(\beta\Delta x/2)$ and $\sin^2(\gamma\Delta y/2)$ are unity [4], so

$$\frac{\psi(t+\Delta t)}{\psi(t)} = (1-4r_1-z) + 2r_1(-2)$$
$$= (1-4r_1-z) - 4r_1 = (1-8r_1-z) = \xi.$$

For stable situation, we need $|\xi| \le 1$, so $-1 \le (1-8r_1-z) \le 1$,

Case 1: $-1 \le (1-8r_1-z) \quad \Rightarrow r_1 \le \frac{2-z}{8}$

Case 2: $(1-8r_1-z) \le 1 \Rightarrow r_1 \ge \frac{-z}{8} \Rightarrow$ neglect this case because $r_1 \ge 0$.

For the second (Linearized) equation of Glycolysis model which is in the form

$$v_{p,q,n+1} = (1-4r_2-Kz)v_{p,q,n} + r_2(v_{p+1,q,n} + v_{p-1,q,n} + v_{p,q+1,n} + v_{p,q-1,n}).$$

Let $V_{p,q,n} = \varphi(t)e^{m\beta x}e^{m\gamma y}$ and substituting it in the above equation to obtain

$$\varphi(t+\Delta t)e^{m\beta x}e^{m\gamma y} = (1-4r_2-Kz)\varphi(t)e^{m\beta(x+\Delta x)}e^{m\gamma y} + r_2(\varphi(t)e^{m\beta(x-\Delta x)}e^{m\gamma y} +$$
$$\varphi(t)e^{m\beta x}e^{m\gamma(y+\Delta y)} + \varphi(t)e^{m\beta x}e^{m\gamma(y-\Delta y)})$$

Dividing both sides of the above equation by $\varphi(t)e^{m\beta x}e^{m\gamma y}$ to obtain

$$\frac{\varphi(t+\Delta t)}{\varphi(t)} = \xi = (1-4r_2-Kz) + m_2[e^{m\beta\Delta x} + e^{-m\beta\Delta x} + e^{m\gamma\Delta y} + e^{-m\gamma\Delta y}]$$
$$= (1-4m_2-Kz) + r_2[2\cos(\beta\Delta x) + 2\cos(\gamma\Delta y)]$$
$$= (1-4r_2-Kz) + 2r_2[1-2\sin^2(\beta\Delta x/2) + 1 - 2\sin^2(\gamma\Delta y/2)]$$
$$= (1-4r_2\sin^2(\beta\Delta x/2) - 4r_2\sin^2(\gamma\Delta y/2) - Kz$$
$$= (1-8r_2-Kz).$$

For some values of β and γ, we can assume that $\sin^2(\beta\Delta x/2)$ and $\sin^2(\gamma\Delta y/2)$ are unity [4], so

$\frac{\varphi(t+\Delta t)}{\varphi(t)} = \xi = (1-8r_2-Kz)$ the equation is stable if $|\xi| \le 1$ which implies that

$|1-8r_2 \text{ - Kz}| \le 1 \Rightarrow -1 < (1-8r_2-Kz) < 1$, which are located in two cases

Case 1: $-1 < (1-8r_2-Kz) \Rightarrow 8r_2 < 2-Kz \Rightarrow r_2 < \frac{2-Kz}{8}$ and

Case 2: $(1-8r_2-Kz) < 1 \Rightarrow 8r_2 \ge -Kz \Rightarrow r_2 \ge -Kz/8$.

So the system is stable under the conditions $r_1 \le \frac{2-z}{8}$,and $r_2 < \frac{2-Kz}{8}$.

2.3.2 Numerical stability of ADI: The ADI finite difference form for the first equation of Glycolysis model is given in the form

$$(1+2r_1)u_{p,q,n+1} = r_1(u_{p+1,q,n+1} + u_{p-1,q,n+1}) + r_2(u_{p,q+1,n} + u_{p,q-1,n}) +$$
$$(1-2r_2-z)u_{p,q,n}. + zKv_{p,q,n} + zu^2_{p,q,n}v_{p,q,n} + \rho z$$

By linearization the term $zu^2_{p,q,n}$ vanishes, and for the same values of K and ρ the terms $zKv_{p,q,n}$, ρz vanishes. Then the equation take the form

$$(1+2r_1)u_{p,q,n+1} = r_1(u_{p+1,q,n+1} + u_{p-1,q,n+1}) + r_2(u_{p,q+1,n} + u_{p,q-1,n}) + (1-2r_2-z)u_{p,q,n}.$$

In order to study the stability of the above equation, after letting $u_{p,q,n} = \psi(t)e^{m\beta x}e^{m\gamma y}$, where $m=\sqrt{-1}$, we obtain

$$(1+2r_1)\psi(t+\Delta t)e^{m\beta x}e^{m\gamma y} = r_1(\psi(t+\Delta t)(e^{m\beta(x+\Delta x)}e^{m\gamma y} + e^{m\beta(x-\Delta x)}e^{m\gamma y}) + r_2(\psi(t)(e^{m\beta x}e^{m\gamma(y+\Delta y)} + e^{m\beta x}e^{m\gamma(y-\Delta y)}) + (1-2r_1-z)\psi(t)e^{m\beta x}e^{m\gamma y}.$$

Dividing both sides of the above equation by $e^{m\beta x}e^{m\gamma y}$ to get

$$(1+2r_1)\psi(t+\Delta t) = r_1(\psi(t+\Delta t)(e^{m\beta\Delta x} + e^{-m\beta\Delta x}) + r_2(\psi(t)(e^{m\gamma\Delta y} - e^{-m\gamma\Delta y}) + (1-2r_1-z)\psi(t).$$

Rearranging, we get

$$= r_1\psi(t+\Delta t))(2\cos(\beta\Delta x) + r_1\psi(t)(2\cos(\gamma\Delta y) + (1+2r_2-z)\psi(t)$$

$$= r_1\psi(t+\Delta t)[2(1-2\sin^2(\beta\Delta x/2)] + r_2\psi(t)[2(1-2\sin^2(\beta\Delta y/2)] + (1-2r_1-k(b+1))\psi(t).$$

For some values of β and γ, assume that $\sin^2(\beta\Delta x/2)$ and $\sin^2(\gamma\Delta y/2)$ are unity [4] , so the equation will take the form

$$(1+2r_1)\psi(t+\Delta t) = r_1\psi(t+\Delta t)(-2) + r_2\psi(t)(-2) + (1-2r_2-z)\psi(t)$$

$$(1+4r_1)\psi(t+\Delta t) = (1-4r_1-z)\psi(t)$$

$$\frac{\psi(t+\Delta t)}{\psi(t)} = \frac{(1-4r_2-z)}{(1+4r_1)} = \xi.$$

So $\dfrac{\psi(t+\Delta t)}{\psi(t)} = \dfrac{1-(4r_2+z)}{1+4r_1} = \xi_1.$

Similarly, for the second equation of the Glycolysis model we assume that $v_{p,q,n} = \varphi(t)e^{m\beta x}e^{m\gamma y}$, where $m=\sqrt{-1}$,the finite difference form of this equation is

$$(1+2r_1)v_{p,q,n+1} = r_1(v_{p+1,q,n+1} + v_{p-1,q,n+1}) + r_2(v_{p+1,q,n+1} + v_{p-1,q,n+1}) + (1-Kz-2r_2)v_{p,q-1,n+1}$$

Then

$$(1+2r_2)\varphi(t+\Delta t)e^{m\beta x}e^{m\gamma y} = r_1\varphi(t+\Delta t)(e^{m\beta(x+\Delta x)}e^{m\gamma y} + e^{m\beta(x-\Delta x)}e^{m\gamma y}) + r_2(\varphi(t)(e^{m\beta x}e^{m\gamma(y+\Delta y)} + e^{m\beta x}e^{m\gamma(y-\Delta y)}) + (1-Kz-2r_2)\varphi(t)e^{m\beta x}e^{m\gamma y}.$$

Dividing both sides of the above equation by $e^{m\beta x}e^{m\gamma y}$, to get

$$(1+2r_2)\varphi(t+\Delta t)e^{m\beta x}e^{m\gamma y} - r_2(\varphi(t+\Delta t)e^{m\beta(x+\Delta x)}e^{m\gamma y} + \varphi(t+\Delta t)e^{m\beta(x-\Delta x)}e^{m\gamma y}) = r_2(\varphi(t)e^{m\beta x}e^{m\gamma(y+\Delta y)} - 2\varphi(t)e^{m\beta x}e^{m\gamma y} + \varphi(t)e^{m\beta x}e^{m\gamma(y-\Delta y)}) + \varphi(t)e^{m\beta x}e^{m\gamma y}.$$

Which implies that

$$(1+2r_1)\varphi(t+\Delta t) = r_1\varphi(t+\Delta t)[2\cos(\gamma\Delta x)] + r_2\varphi(t)[2\cos(\gamma\Delta y)] + (1-Kz-2r_2)\varphi(t)$$

$$(1+2r_1)\varphi(t+\Delta t) = r_1\varphi(t+\Delta t)[2(1-2\sin^2(\beta\Delta x/2)] + r_2(2(1-2\sin^2(\gamma\Delta y/2)\varphi(t) + (1-Kz-2r_2)\varphi(t).$$

For some values of β and γ, we have $\sin^2(\beta\Delta x/2)$ and $\sin^2(\gamma\Delta y/2)$ are unity, so we have

$$(1+2r_1)\varphi(t+\Delta t) = r_1\varphi(t+\Delta t)(-2) + r_2\varphi(t) + (1-Kz-2r_2)\varphi(t).$$

$$(1+4r_1)\varphi(t+\Delta t) = (1-Kz-4r_2)\varphi(t)$$

$$\frac{\psi(t+\Delta t)}{\psi(t)} = \frac{(1-(4r_2+Kz)}{(1+4r_1)} = \xi_2.$$

Where ξ_1. and ξ_2. stand for the I-plane and II-plane respectively, each of the above terms is conditionally stable. However the combined two-levels has the form

$$\xi_{ADI} = \xi_1\xi_2 = [\frac{(1-(4r_2+Kz))}{(1+4r_1)}][\frac{(1-(4r_2+Kz))}{(1+4r_1)}].$$

Thus the above scheme is unconditionally stable, each individual equation is conditionally stable by itself, and combined two-level is completely stable.

III. APPLICATION (NUMERICAL EXAMPLE)

We solved the following example numerically to illustrate the efficiency of the presented methods, suppose we have the system

$$\frac{\partial u}{\partial t} = d_1 \Delta u - u + kv + u^2 v + \rho,$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - kv - u^2 v + \delta$$

We the initial conditions

$U(x, 0) = Us + 0.01 \sin(\pi x / L)$ f

or $0 \leq x \leq L$

$V(x, 0) = Vs - 0.12 \sin(\pi x / L)$ for $0 \leq x \leq L$

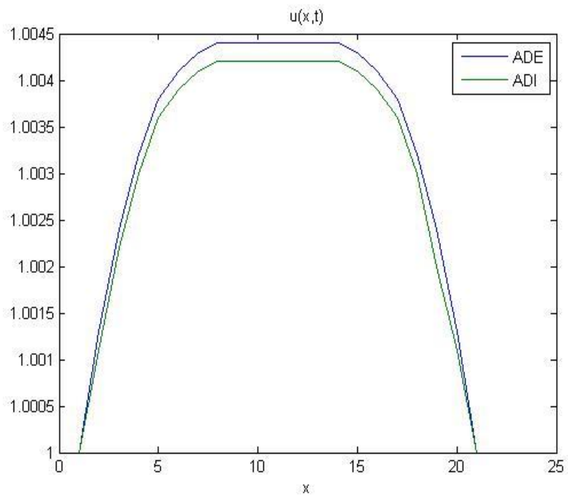
$U(0, t) = Us$, $U(L, t) = Us$ and $V(0, t) = Vs$, $V(L, t) = Vs$

We will take

$d_1 = d_2 = 0.01$, $\rho = 0.09$, $\delta = -0.004$, $U_s = 0$, $V_s = 1$

Then the results in more details are shown in following table and figure:

1.0003	1.0000
1.0017	1.0013
1.0028	1.0024
1.0036	1.0032
1.0040	1.0038
1.0042	1.0041
1.0042	1.0043
1.0041	1.0044
1.0040	1.0044
1.0038	1.0044
1.0038	1.0044
1.0038	1.0044
1.0040	1.0044
1.0041	1.0044
1.0042	1.0043
1.0042	1.0041
1.0040	1.0038
1.0036	1.0032
1.0028	1.0024
1.0017	1.0013
1.0003	1.0000



IV. CONCLUSION

We saw that alternating direction implicit is more accurate than alternating direction explicit method for solving Glycolysis model and we found that ADE method is stable under condition $r_1 \leq \frac{2-z}{8}$, and $r_1 \leq \frac{2-Kz}{8}$, while ADI is unconditionally stable.

REFERENCES

[1]. Ames,W.F.(1992);'Numerical Methods for Partial Differential Equations'3rd ed.**Academic, Inc**. Dynamics of PDE, Vol.4, No.2, 167-196, 2007.

[2]. Ellbeck, J.C.(1986)" The Pseude-Spectral Method and Path Following Reaction –Diffusion Bifurcation Studies"**SIAM J.Sci.Stat.Comput**.,Vol. 7,No.2,Pp.599-610.

[3]. Logan,J.D.,(1987),"Applied Mathematics", **John Wiley and Sons.**

[4]. Mathews, J. H. and Fink, K. D., (1999)" numerical methods using Matlab", Prentice- Hall, Inc.

[5]. Shanthakumar, M.;(1989)"Computer Based Numerical Analysis "**Khana Publishers.**

[6]. Sherratt,J.A.,(1996) "Periodic Waves in Reaction Diffusion Models of Oscillatory Biological Systemd; **Forma**, 11,61-80 **.**

[7]. Smith, G., D., Numerical Solution of Partial Differential Equations, Finite Difference Methods, 2nd edition, Oxford University Press, (1965).

[8]. Temam R., Infinite dimensional Dynamical systems in chanics and physics (Springer-Verlag,berlin,1993), D.Barkley and I.G.Kevrekidis, Chaos 4,453 (1994)

[9]. Yuncheng You , Global Dynamics of the Brusselator Equations,Dynamics of PDE, Vol.4, No.2 (2007), 167-196.