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# Periodic Solutions for Nonlinear Systems of Integro-Differential Equations of Operators with Impulsive Action

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**Abstract:-** In this paper we investigate the existence and approximation of the construction of periodic solutions for nonlinear systems of integro-differential equation of operators with impulsive action, by using the numerical-analytic method for periodic solutions which is given by Samoilenko. This investigation leads us to the improving and extending to the above method and expands the results gained by Butris.

**Keyword and Phrases:-** Numerical-analytic methods existence of periodic solutions, nonlinear integro-differential equations, subjected to impulsive action of operators.

## I. INTRODUCTION

They are many subjects in mechanics, physics, biology, and other subjects in science and engineering requires the investigation of periodic solution of integro-differential equations with impulsive action of various types and their systems in particular, they are many works devoted to problems of the existence of periodic solutions of integro-differential equations with impulsive action . samoilenko[5,6] assumed a numerical analytic method to study to periodic solution for ordinary differential equation and this method include uniformly sequences of periodic functions' as Intel studies [2,3,4,7,8].

The present work continues the investigations in this direction. These investigations lead us to the improving and extending the above method and expand the results gained by Butris [1].

$$\frac{dx}{dt} = f(t, x, Ax, Bx, \int_{t-T}^{t} g(t, x(s), Ax(s), Bx(s))ds), t \neq t_i$$

$$\Delta x|_{t=t_i} = I_i(x, Ax, Bx, \int_{t-T}^{t} g\big(s, x(s), Ax(s), Bx(s)\big)ds) \qquad \cdots (1.1)$$

Where $x \in D \subset R^n$, $D$ is closed bounded domain.

The vector functions $f(t, x, y, z, w)$ and $g(t, x, y, z)$ are defined on the domain

$$(t, x, y, z, w) \in R^1 \times D \times D_1 \times D_2 \times D_3 = (-\infty, \infty) \times D \times D_1 \times D_2 \times D_3 \quad \cdots (1.2)$$

Which are piecewise continuous functions in $t, x, y, z, w$ and periodic in $t$ of period $T$, where $D_1, D_2$ and $D_3$ are bounded domain subset of Euclidean spaces $R^m$.

Let $I_i(x, y, z, w)$ be continuous vector functions, defined on the domain (1.2) and

$$I_{i+p}(x, y, z, w) = I_i(x, y, z, w)\, , t_{i+p} = t_i + T \qquad \cdots (1.3)$$

For all $i \in z^+, x \in D, y \in D_1, z \in D_2, w \in D_3$ and for some number $p$ .

Let the operators A and B transform any piecewise continuous functions from the domain $D$ to the piecewise contiuous function in the domain $D_1$ and $D_2$ respectively. Moreover $Ax(t + T) = Ax(t)$ and $Bx(t + T) = Bx(t)$.

Suppose that $f(t, x, y, z, w), g(t, x, y, z), I_i(t, x, y, z, w)$ and the operators A, B satisfy the following inequalities:

$$\|f(t, x, y, z, w)\| \leq M, \|I_i(t, x, y, z, w)\| \leq N, \qquad \cdots (1.4)$$

$$\|f(t, x_1, y_1, z_1, w_1) - f(t, x_2, y_2, z_2, w_2)\| \leq K(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\| + \|w_1 - w_2\|)$$

$$\|g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2)\| \leq Q(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|) \qquad \cdots (1.5)$$

And

$$\|I_i(t, x_1, y_1, z_1, w_1) - I_i(t, x_2, y_2, z_2, w_2)\| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\| + \|w_1 - w_2\|) \qquad \cdots (1.6)$$

$$\left.\begin{aligned}\|Ax_1(t)-Ax_2(t)\| &\le G\|x_1(t)-x_2(t)\| \\ \|Bx_1(t)-Bx_2(t)\| &\le H\|x_1(t)-x_2(t)\|\end{aligned}\right\} \quad \cdots (1.7)$$

For all $t \in R^1$, $x, x_1, x_2 \in D, y, y_1, y_2 \in D_1, z, z_1, z_2 \in D_2$ and $w, w_1, w_2 \in D_3$ where $M, N, K, Q, L$ and $G, H$ are a positive constants.
Consider the matrix

$$\Lambda = \begin{pmatrix} K_1 \frac{T}{3} & K_1 \\ pTK_2 & 2pK_2 \end{pmatrix} \quad \cdots (1.8)$$

Where

$$K_1 = K[1+G+H+Q(1+G+H)], K_2 = L[1+G+H+Q(1+G+H)]$$

We define the non-empty sets as follows

$$\left.\begin{aligned} D_f &= D - M\frac{T}{2}(1+\frac{4p}{T}) \\ D_{1f} &= D_1 - GM\frac{T}{2}(1+\frac{4p}{T})\end{aligned}\right\} \quad \cdots (1.9)$$

and

$$\left.\begin{aligned} D_{2f} &= D_2 - HM\frac{T}{2}(1+\frac{4p}{T}) \\ D_{3f} &= D_3 - QM\frac{T^2}{2}(1+\frac{4p}{T})\end{aligned}\right\} \quad \cdots (1.10)$$

Furthermore, we suppose that the greatest Eigen-value $\lambda_{max}$ of the matrix $\Lambda$ does not exceed unity, i.e.

$$\Lambda = K_1\frac{T}{2} + pK_2(2+K_1\frac{T}{2}) < 1 \quad \cdots (1.11)$$

**Lemma 2.1.** Let $f(t)$ be a continuous (piecewise continuous) vector function in the interval $0 \le t \le T$. Then

$$\left\| \int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds \right\| \le \alpha(t) \max_{t\in[0,T]} \|f(t)\|,$$

Where $\alpha(t) = 2t(1-\frac{t}{T})$ . (For the proof see [3]) .
Assume that
$Ax_m(t,x_0) = y_m(t,x_0)$ , $Bx_m(t,x_0) = z_m(t,x_0)$ and

$$w_m(t,x_0) = \int_{t-T}^{t} g(s, x_m(s,x_0), Ax_m(s,x_0), Bx_m(s,x_0))ds \, ,$$

$m = 0,1,2,\cdots$

## II. APPROXIMATE SOLUTION

The investigation of a periodic approximate solution of the system (1.1) makes essential use of the statements and estimates given below.

**Theorem 2.1.** If the system of integro-differential equations with impulsive action(1.1) satisfy the inequalities (1.3) to (1.7) and the conditions (1.8), (1.10) has a periodic solution $x = x(t,x_0)$ , passing through the point $(0,x_0), x_0 \in D_f$ ,
$Ax_0 \in D_{1f}$ and $Bx_0 \in D_{2f}$ , then the sequence of functions:

$$\begin{aligned} x_m(t,x_0) = x_0 + \int_0^t [f(s, x_m(s,x_0), y_m(s,x_0), z_m(s,x_0), w_m(s,x_0)) - \\ -\frac{1}{T}\int_0^T g(s, x_m(s,x_0), y_m(s,x_0), z_m(s,x_0), w_m(s,x_0))\, ds]ds + \\ + \sum_{0<t_i<t} I_i\,(x_m(t_i,x_0), y_m(t_i,x_0), z_m(t_i,x_0), w_m(t_i,x_0)) - \\ -\frac{t}{T}\sum_{i=1}^{p} I_i(x_m(t_i,x_0), y_m(t_i,x_0), z_m(t_i,x_0), w_m(t_i,x_0)) \end{aligned} \quad \cdots (2.1)$$

With
$x_0(t,x_0) = x_0$ , $m = 0,1,2,\cdots$

is periodic in $t$ of period $T$, and uniformly convergent as $m \to \infty$ in

$$(t, x_0) \in R^1 \times D_f = (-\infty, \infty) \times D_f \qquad \cdots (2.2)$$

To the vector function $x^0(t, x_0)$ defined on the domain (1.2), which is periodic in $t$ of period $T$ and satisfying the system of integral equation

$$x(t, x_0) = x_0 + \int_0^t [f(s, x(s, x_0), y(s, x_0), z(s, x_0), w(s, x_0)) - -\frac{1}{T}\int_0^T g(s, x(s, x_0), y(s, x_0), z(s, x_0), w(s, x_0))\, ds]ds + + \sum_{0<t_i<t} I_i\,(x(t_i, x_0), y(t_i, x_0), z(t_i, x_0), w(t_i, x_0)) - -\frac{t}{T}\sum_{i=1}^{p} I_i(x(t_i, x_0), y(t_i, x_0), z(t_i, x_0), w(t_i, x_0)) \qquad \cdots (2.3)$$

Which a unique solution of the system (1.1) provided that

$$\|x^0(t, x_0) - x_0\| \le \frac{MT}{2}\,(1 + \frac{4p}{T}) \qquad \cdots (2.4)$$

and

$$\|x^0(t, x_0) - x_m(t, x_0)\| \le \lambda^m (1 - \lambda)^{-1} M \frac{T}{2}(1 + \frac{4p}{T}) \qquad \cdots (2.5)$$

for all $m \ge 1, t \in R^1$, where the eigen-value $\lambda$ of the matrix $\Delta$ is a positive fraction less than one.

**Proof.** Consider the sequence of functions $x_1(t, x_0), x_2(t, x_0), \cdots, x_m(t, x_0), \cdots$,
defined by recurrence relation (2.1). Each of the functions of the sequence is periodic in $t$ of period $T$.
Now, by Lemma 1.1, we have

$$\|x_m(t, x_0) - x_0\| = \|x_0 + \int_0^t [f(s, x_m(s, x_0), y_m(s, x_0), z_m(s, x_0), w_m(s, x_0)) - -\frac{1}{T}\int_0^T g(s, x_m(s, x_0), y_m(s, x_0), z_m(s, x_0), w_m(s, x_0))\, ds]ds + + \sum_{0<t_i<t} I_i\,(x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0)) - -\frac{t}{T}\sum_{i=1}^{p} I_i(x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0)) - x_0\| \le$$

$$\le (1 - \frac{t}{T})\int_0^t \|f(s, x_m(s, x_0), y_m(s, x_0), z_m(s, x_0), w_m(s, x_0))\|\, ds + +\frac{t}{T}\int_t^T \|f(s, x_m(s, x_0), y_m(s, x_0), z_m(s, x_0), w_m(s, x_0))\| ds + + \sum_{0<t_i<t} I_i\,\|(x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0))\| + +\frac{\|t\|}{T}\sum_{i=1}^{p}\left\|I_i\left(x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0)\right)\right\| \le$$

$$\le \alpha(t)M + 2pM$$

$$\le M\frac{T}{2}(1 + \frac{4p}{T}) \qquad \cdots (2.6)$$

From (1.7) and (2.4), we have

$$\left.\begin{aligned} \|Ax_m(t, x_0) - Ax_0\| &\le QM\frac{T}{2}(1 + \frac{4p}{T}) \\ \|Bx_m(t, x_0) - Bx_0\| &\le HM\frac{T}{2}(1 + \frac{4p}{T}) \end{aligned}\right\} \qquad \cdots (2.7)$$

for all $x_0 \in D_f$.
Also

$$\|w_m(t,x_0) - w_0(t,x_0)\| = \left\| \int_{t-T}^{t} g\big(s, x_m(s,x_0), Ax_m(s,x_0), Bx_m(s,x_0)\big)ds - \right.$$

$$\left. - \int_{t-T}^{t} g(s, x_0, Ax_0, Bx_0)ds \right\| \le$$

$$\le \int_{t-T}^{t} K(\|x_m(s,x_0) - x_0\| + \|Ax_m(s,x_0) - Ax_0\| + \|Bx_m(s,x_0) - Bx_0\|)ds$$

$$\le \int_{t-T}^{t} [K(M\frac{T}{2}(1+\frac{4p}{T}) + GM\frac{T}{2}(1+\frac{4p}{T}) + HM\frac{T}{2}(1+\frac{4p}{T}))]ds$$

$$= KM\frac{T}{2}(1+\frac{4p}{T})(1+G+H) \qquad \cdots (2.8)$$

For $m = 1$, in (2.1), we get

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le (1-\frac{t}{T})\int_0^t K(\|x_1(s,x_0) - x_0\| + \|y_1(s,x_0) - y_0(s,x_0)\|$$

$$+\|z_1(s,x_0) - z_0(s,x_0)\| + \|w_1(s,x_0) - w_0(s,x_0)\|)ds +$$

$$+\frac{t}{T}\int_t^T K(\|x_1(s,x_0) - x_0\| + \|y_1(s,x_0) - y_0(s,x_0)\| +$$

$$+\|z_1(s,x_0) - z_0(s,x_0)\| + \|w_1(s,x_0) - w_0(s,x_0)\|)ds +$$

$$+ \sum_{0<t_i<t} L\,\|x_1(t_i,x_0) - x_0\| + \|y_1(t_i,x_0) - y_0(t_i,x_0)\| +$$

$$+\|z_1(t_i,x_0) - z_0(t_i,x_0)\| + \|w_1(t_i,x_0) - w_0(t_i,x_0)\|) +$$

$$+\frac{\|t\|}{T}\sum_{i=1}^{p} L\,(\|x_1(t_i,x_0) - x_0\| + \|y_1(t_i,x_0) - y_0(t_i,x_0)\| +$$

$$+\|z_1(t_i,x_0) - z_0(t_i,x_0)\| + \|w_1(t_i,x_0) - w_0(t_i,x_0)\|) +$$

$$\le (1-\frac{t}{T})\int_0^t [K(1+G+H+Q(L+G+H))\,M\frac{T}{2}(1+\frac{4p}{T})]ds +$$

$$+\frac{t}{T}\int_t^T [K(1+G+H+Q(L+G+H))\,M\frac{T}{2}(1+\frac{4p}{T})]ds +$$

$$+[L(1+G+H+Q(L+G+H))]M\frac{T}{2}(1+\frac{4p}{T}) \le$$

$$\le \alpha(t)[K(1+G+H+Q(L+G+H))]M\frac{T}{2}(1+\frac{4p}{T}) +$$

$$+LpMT(1+G+H+Q(L+G+H))(1+\frac{4p}{T}) \le$$

$$\le N_1\alpha(t) + M_1 = N_1\frac{T}{2} + M_1 \qquad \cdots (2.10)$$

If the following inequality is true

$$\|x_m(t,x_0) - x_{m-1}(t,x_0)\| \le N_{m-1}\alpha(t) + M_{m-1}$$

$$\le N_{m-1}\frac{T}{2} + M_{m-1} \qquad \cdots (2.11)$$

for all $m = 1,2,\cdots$ .
then, we shall to prove that

$$\|x_{m+1}(t,x_0) - x_m(t,x_0)\| \le K_1(N_{m-1}\frac{T}{2} + M_{m-1})\alpha(t) + 2pK_2(N_{m-1}\frac{T}{2} + M_{m-1}) \qquad \cdots (2.12)$$

for all $m = 0,1,2,\cdots$ .

By mathematical induction, we have

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le N_m\alpha(t)+M_m \le \frac{T}{2}N_m+M_m \ , \qquad \cdots(2.13)$$

Where

$$\left.\begin{aligned} N_{m+1} &= K_1\frac{T}{2}N_m+K_1M_m \quad , \\ M_{m+1} &= pK_2TN_m+2pK_2M_m \quad , \quad N_0=M \ , \quad M_0=2pM \ . \end{aligned}\right\} \qquad \cdots(2.14)$$

$m=0,1,2,\cdots$.

It is sufficient to show that all solutions of (2.14) approach zero as $m\to\infty$, i.e. it is necessary and sufficient that the eigen values of the matrix $\Lambda$ are assumed to be within a unit circle.

It is well-known that the characteristic equation of the matrix $\Lambda$ is

$$K_1\frac{T}{2}+pK_2(2+K_1\frac{T}{2})<1 \qquad \cdots(2.15)$$

and this ensures that the sequence of functions (2.1) is convergent uniformly on the domain (2.2) as $m\to\infty$ .

Let

$$\lim_{m\to\infty} x_m(t,x_0)=x_\infty(t,x_0) \qquad \cdots(2.13)$$

Since the sequence of functions (2.2) is periodic in $t$ of period $T$, then the limiting is also periodic in $t$ of period $T$.

Moreover, B lemma 1.1 and (2.15) and the following inequality

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le \sum_{i=0}^{k-1}\|x_{m+i+1}(t,x_0)-x_{m+i}(t,x_0)\| \le$$

$$\le \sum_{i=0}^{k-1}\lambda^{m+i}\,M\frac{T}{2}(1+\frac{4p}{T})$$

the inequalities (2.4) and (2.5) are satisfied for all $m\ge 0$ .

Finally, we have to show that $x(t,x_0)$ is unique solution of (1.1). on the contrary, we suppose that there is at least two different solutions $x(t,x_0)$ and $r(t,x_0)$ of (1.1).

From (2.3), the following inequality holds:

$$\|x(t,x_0)-r(t,x_0)\| \le (1-\frac{t}{T})\int_0^t K_1\,\|x(s,x_0)-r(s,x_0)\|ds+$$

$$+\frac{t}{T}\int_t^T K_1\|x(s,x_0)-r(s,x_0)\|\,ds+\sum_{0<t_i<t}K_2\,\|x(t_i,x_0)-r(t_i,x_0)\|+$$

$$+\frac{t}{T}\sum_{i=1}^{p}K_2\,\|x(t_i,x_0)-r(t_i,x_0)\| \qquad \cdots(2.16)$$

Setting $\|x(t,x_0)-r(t,x_0)\|=h(t)$, the inequality (2.16) can be written as:

$$h(t)\le(1-\frac{t}{T})\int_0^t K_1\,h(s)ds+\frac{t}{T}\int_t^T K_1h(s)\,ds+\sum_{0<t_i<t}K_2\,h(t_i)+\frac{t}{T}\sum_{i=1}^{p}K_2\,h(t_i)$$

Let $\max_{t\in[0,T]}h(t)=h_0\ge 0$ . By iteration, we get:

$$h(t)\le N_m\alpha(t)+M_m \qquad \cdots(2.17)$$

From (2.14), we have:

$$\begin{pmatrix}N_{m+1}\\ M_{m+1}\end{pmatrix}=\begin{pmatrix}K_1\frac{T}{2} & K_1\\ pTK_2 & 2pK_2\end{pmatrix}\begin{pmatrix}N_m\\ M_m\end{pmatrix} \qquad \cdots(2.18)$$

which satisfies the initial conditions $N_0=0, M_0=h_0$ That is

$$\begin{pmatrix}N_m\\ M_m\end{pmatrix}=\begin{pmatrix}K_1\frac{T}{2} & K_1\\ pTK_2 & 2pK_2\end{pmatrix}^m\begin{pmatrix}0\\ h_m\end{pmatrix} \qquad \cdots(2.19)$$

Hence it is clear that if the condition (2.15) is satisfied then $N_m\to 0$ and $M_m\to 0$ as $m\to\infty$.

From the relation (2.17) we get $h(t)\equiv 0$ or $x(t,x_0)=r(t,x_0)$, i.e. $x(t,x_0)$ is a unique solution of (1.1). ■

## III. EXISTENCE OF SOLUTION

The problem of the existence of periodic solution of period $T$ of the system (1.1) is uniquely connected with the existence of zeros of the function $\Delta(x_0)$ which has the form

$$\Delta(x_0)=\frac{1}{T}[\int_0^T f(t,x^0(t,x_0),y^0(t,x_0),z^0(t,x_0),w^0(t,x_0))\,dt+ \\ +\sum_{i=1}^{p} I_i\,(x^0(t_i,x_0),y^0(t_i,x_0),z^0(t_i,x_0),w^0(t_i,x_0))], \qquad \cdots(3.1)$$

since this function is approximately determined from the sequence of functions

$$\Delta_m(x_0)=\frac{1}{T}[\int_0^T f(t,x_m(t,x_0),y_m(t,x_0),z_m(t,x_0),w_m(t,x_0))\,dt+ \\ +\sum_{i=1}^{p} I_i\,(x_m(t_i,x_0),y_m(t_i,x_0),z_m(t_i,x_0),w_m(t_i,x_0))] \qquad \cdots(3.2)$$

where $x^0(t,x_0)$ is the limiting of the sequence of functions (2.1). Also
$y^0(t,x_0)=Ax^0(t,x_0),\ z^0(t,x_0)=Bx^0(t,x_0)$ and

$$w^0(t,x_0)=\int_{t-T}^{t} g(s,x^0(s,x_0),Ax^0(s,x_0),Bx^0(s,x_0))\,ds\,.$$

Now, we prove the following theorem taking in to ///// that the following inequality will be satisfied for all $m\geq 1$.

$$\|\Delta(x_0)-\Delta_m(x_0)\|\leq \lambda^m(1-\lambda)^{-1}(K_1+\frac{p}{T}K_2)\frac{MT}{2}(1+\frac{4p}{T}) \qquad \cdots(3.3)$$

**Theorem 3.1.** If the system of equations (1.1) satisfies the following conditions:
$(i)$ the sequence of functions (3.2) has an isolated singular point $x_0=x^0,\ \Delta_m(x_0)\equiv 0$, for all $t\in R^1$;
$(ii)$ the index of this point is nonzero;
$(iii)$ there exists a closed convex domain $D_4$ belonging to the domain $D_f$ and possessing a unique singular point $x^0$ such that on it is boundary $\Gamma_{D_4}$ the following inequality holds

$$\inf_{x\in\Gamma_{D_4}}\|\Delta_m(x^0)\|\leq \lambda^m(1-\lambda)^{-1}(K_1+\frac{p}{T}K_2)\frac{MT}{2}(1+\frac{4p}{T})$$

for all $m\geq 1$. Then the system (1.1) has a periodic solution $x=x(t,x_0)$ for which $x(0)\in D_4$.
**Proof.** By using the inequality (3.3) we can prove the theorem is a similar way to that of theorem 2.1.2 [2].

**Remark 3.1.** When $R^n=R^1$, i.e. when $x^0$ is a scalar, theorem 3.1 can be proved by the following.
**Theorem 3.2.** Let the functions $f(t,x,y,y,z,w)$ and $I_i(x,y,z,w)$ of system (1.1) are defined on the interval $[a,b]$ in $R^1$. Then the function (3.2) satisfies the inequalities:

$$\left.\begin{aligned} \min_{a+\frac{MT}{2}\left(1+\frac{4p}{T}\right)\leq x^0\leq b-\frac{MT}{2}\left(1+\frac{4p}{T}\right)} \Delta_m(x^0) &\leq -\lambda^m(1-\lambda)^{-1}(K_1+\frac{p}{T}K_2) \\ \max_{a+\frac{MT}{2}\left(1+\frac{4p}{T}\right)\leq x^0\leq b-\frac{MT}{2}\left(1+\frac{4p}{T}\right)} \Delta_m(x^0) &\geq \lambda^m(1-\lambda)^{-1}(K_1+\frac{p}{T}K_2) \end{aligned}\right\} \qquad \cdots(3.4)$$

Then (1.1) has a periodic solution in $t$ of period $T$ for which

$$x^0(0)\in[a+\frac{MT}{2}(1+\frac{4p}{T}),b-\frac{MT}{2}(1+\frac{4p}{T})]\,.$$

**Proof.** Let $x_1$ and $x_2$ be any points of the interval
$[a+\frac{MT}{2}(1+\frac{4p}{T}),b-\frac{MT}{2}(1+\frac{4p}{T})]$ such that

$$\left.\begin{aligned} \Delta_m(x_1) &= \min_{a+\frac{MT}{2}\left(1+\frac{4p}{T}\right)\leq x^0\leq b-\frac{MT}{2}\left(1+\frac{4p}{T}\right)} \Delta_m(x^0)\ , \\ \Delta_m(x_2) &= \max_{a+\frac{MT}{2}\left(1+\frac{4p}{T}\right)\leq x^0\leq b-\frac{MT}{2}\left(1+\frac{4p}{T}\right)} \Delta_m(x^0)\ . \end{aligned}\right\} \qquad \cdots(3.5)$$

By using the inequalities (3.3) and (3.4) , we have

$$\left.\begin{aligned} \Delta_m(x_1) &= \Delta_m(x_1)+\left(\Delta_m(x_1)-\Delta_m(x_1)\right)<0\ , \\ \Delta_m(x_2) &= \Delta_m(x_2)+(\Delta_m(x_2)-\Delta_m(x_2))>0\ . \end{aligned}\right\} \qquad (3.6)$$

The continuity of $\Delta(x^0)$ and by using (3.6), there exists a point $x^0, x^0 \in [x_1, x_2]$, such that $\Delta(x^0) \equiv 0$, i.e. $x = x(t, x_0)$ is a periodic solution in $t$ of period $T$ for which $x^0(0) \in [a + \frac{MT}{2}(1 + \frac{4p}{T}), b - \frac{MT}{2}(1 + \frac{4p}{T})]$. ■

Similar results can be obtained for other class of integro-differential equations of operators with impulsive action.

In particular, the system of integro-differential equations which has the form

$$\frac{dx}{dt} = f(t, x, Ax, Bx, \int_{a(t)}^{b(t)} g\big(s, x(s), Ax(s), Bx(s)\big)ds,$$

$$, \int_{-\infty}^{t} F(t,s)\, g(s, x(s), Ax(s), Bx(s)ds), \qquad t \neq t_i$$

$$\Delta x = |_{t=t_i}\, I_i(x, Ax, Bx, \int_{a(t_i)}^{b(t_i)} g\big(s, x(s), Ax(s), Bx(s)\big)ds,$$

$$, \int_{-\infty}^{t_i} F(t_i, s) g(s, x(s), Ax(s), Bx(s))ds) \qquad \cdots (3.7)$$

In this system (3.7), let the vector functions $f(t, x, y, z, w), g(t, x, y, z)$ and scalar functions $a(t), b(t)$ are periodic in $t$ of period $T$, defined and continuous on the domain.

$$(t, x, y, z, w, h) \in R^1 \times G \times G_1 \times G_2 \times G_3 \times G_4 =$$
$$= (-\infty, \infty) \times G \times G_1 \times G_2 \times G_3 \times G_4 \cdots (3.8)$$

Let $I_i(x, y, z, w, h)$ be a continuous vector function which are defined on the domain (3.8). A matrix $F(t, s)$ is defined and continuous in $R^1 \times R^1$ and satisfies the condition $F(t + T, s + T) = F(t, s)$ which $\|F(t, s)\| \leq \delta\, e^{-\gamma(t-s)}$,

$0 \leq s \leq \mathrm{t} \leq \mathrm{T}$, where $\delta, \gamma$ are a positive constants. Suppose that the functions $f(t, x, y, z, w, h)$ and $g(t, x, y, z), I_i(x, y, z, w, h)$ are satisfying the following inequalities:

$$\|f(t, x, y, z, w, h)\| \leq M,\ \|g(t, x, y, z)\| \leq N, \qquad \cdots (3.9)$$

$$\|f(t, x_1, y_1, z_1, w_1, h_1) - f(t, x_2, y_2, z_2, w_2, h_2)\| \leq K^*(\|x_1 - x_2\| + \|y_1 - y_2\| +$$
$$+\|z_1 - z_2\| + \|w_1 - w_2\| + \|h_1 - h_2\|);$$
$$\|g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2)\| \leq Q^*(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|);$$
$$\cdots (3.10)$$
$$\|I_i(x_1, y_1, z_1, w_1, h_1) - I_i(x_2, y_2, z_2, w_2, h_2)\| \leq L^*(\|x_1 - x_2\| + \|y_1 - y_2\| +$$
$$+\|z_1 - z_2\| + \|w_1 - w_2\| + \|h_1 - h_2\|);$$
$$\cdots (3.11)$$

and

$$\left.\begin{array}{l} \|Ax_1(t) - Ax_2(t)\| \leq G^*\|x_1(t) - x_2(t)\|, \\ \|Bx_1(t) - Bx_2(t)\| \leq H^*\|x_1(t) - x_2(t)\|. \end{array}\right\} \qquad \cdots (3.12)$$

for all $t \in R^1, x, x_1, x_2 \in G, y, y_1, y_2 \in G_1, z, z_1, z_2 \in G_2, w, w_1, w_2 \in G_3$, $h, h_1, h_2 \in G_4$, Provided that

$Ax(t + T) = Ax(t), Bx(t + T) = Bx(t), I_{i+p}(x, y, z, w, h) = I_i(x, y, z, w, h)$ and $t_{i+p} = t_i + T$.

Define a non-empty sets as follows:

$$\left.\begin{array}{l} G_f = G - \dfrac{M^*T}{2}(1 + \dfrac{4p}{T}),\ G_{1f} = G_1 - G^*\dfrac{M^*T}{2}(1 + \dfrac{4p}{T}), \\ G_{2f} = G_3 - H^*\dfrac{M^*T}{2}(1 + \dfrac{4p}{T}),\ G_{4f} = G_4 - \beta Q^*\dfrac{M^*T}{2}(1 + \dfrac{4p}{T}), \\ G_{5f} = G_5 - \dfrac{\delta}{\gamma}Q^*\dfrac{M^*T}{2}(1 + \dfrac{4p}{T}). \end{array}\right\} \qquad \cdots (3.13)$$

Furthermore, the largest Eigen-value of $\gamma_{max}$ of the matrix

$$\Gamma = \begin{pmatrix} K_1\dfrac{T}{2} & K_1 \\ pK_2T & 2pK_2 \end{pmatrix}$$

is less than unity, i.e.

$$\frac{1}{2}[K_1\frac{T}{3}+2pK_2+\sqrt{(\frac{K_1T}{3}+2pK_2)^2+\frac{4pK_1K_2T}{3}}]<1\,, \qquad \cdots(3.14)$$

where $K_1=K^*[1+G^*+H^*+\frac{\delta}{\gamma}Q^*+Q^*(1+G^*+\beta)]$,
$K_2=L^*[1+G^*+H^*+\frac{\delta}{\gamma}Q^*+Q^*(1+G^*+\beta)]$ and $\beta=\max_{0\le t\le T}|b(t)-a(t)|$.

**Theorem 3.2.** If the system of integro-differential equations with impulsive action (3.7) satisfying the above assumptions and conditions has a periodic solution $x=\psi(t,x_0)$, then there exists a unique solution which is the limit function of a uniformly convergent sequence which has the form

$$x_m(t,x_0)=x_0+\int_0^t[f(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0),w_m(s,x_0),h_m(s,x_0))-$$
$$-\frac{1}{T}\int_0^T f(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0),w_m(s,x_0),h_m(s,x_0))\,ds]ds+$$
$$+\sum_{0<t_i<t} I_i\,(x_m(t_i,x_0),y_m(t_i,x_0),z_m(t_i,x_0),w_m(t_i,x_0),h_m(t_i,x_0))-$$
$$-\frac{t}{T}\sum_{i=1}^{p} I_i(x_m(t_i,x_0),y_m(t_i,x_0),z_m(t_i,x_0),w_m(t_i,x_0),h_m(t_i,x_0)) \qquad \cdots(3.15)$$

with
$x_0(t,x_0)=x_0\,,\ \ m=0,1,2,\cdots$.
The proof is a similar to that of theorem 1.1.
If we consider the following mapping

$$\Delta_m(x_0)=\frac{1}{T}[\int_0^T f(t,x_m(t,x_0),y_m(t,x_0),z_m(t,x_0),w_m(t,x_0),h_m(t,x_0))\,dt+$$
$$+\sum_{i=1}^{p} I_i\,(x_m(t_i,x_0),y_m(t_i,x_0),z_m(t_i,x_0),w_m(t_i,x_0),h_m(t_i,x_0))]. \qquad \cdots(3.16)$$

Then we can state a theorem similar to theorem 2.2, provided that $\gamma_{max}=\frac{1}{2}[K_1\frac{T}{3}+2pK_2+\sqrt{(\frac{K_1T}{3}+2pK_2)^2+\frac{4pK_1K_2T}{3}}]<1$,

**Remark 3.2.** It is clear that when we put $I_i\equiv 0$, we get a periodic solution for the systems (1.1) and (3.7) without introducing impulsive action.

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