Periodic Solutions for Nonlinear Systems of Integro-Differential Equations of Operators with Impulsive Action

Dr. Raad N. Butris
Assistant Prof. Department of Mathematics Faculty of Science University of Zakho
Iraq-Duhok-Kurdistan Region

Abstract:- In this paper we investigate the existence and approximation of the construction of periodic solutions for nonlinear systems of integro-differential equation of operators with impulsive action, by using the numerical-analytic method for periodic solutions which is given by Samoilenko. This investigation leads us to the improving and extending to the above method and expands the results gained by Butris.

Keyword and Phrases:- Numerical-analytic methods existence of periodic solutions, nonlinear integro-differential equations, subjected to impulsive action of operators.

I. INTRODUCTION

They are many subjects in mechanics, physics, biology, and other subjects in science and engineering requires the investigation of periodic solution of integro-differential equations with impulsive action of various types and their systems in particular, they are many works devoted to problems of the existence of periodic solutions of integro-differential equations with impulsive action. Samoilenko[5,6] assumed a numerical analytic method to study to periodic solution for ordinary differential equation and this method include uniformly sequences of periodic functions’ as Intel studies [2,3,4,7,8].

The present work continues the investigations in this direction. These investigations lead us to the improving and extending the above method and expand the results gained by Butris [1].

\[
\frac{dx}{dt} = f(t, x, A(x), B(x), \int_{t-T}^{t} g(t, x(s), A(x(s)), B(x(s))) ds, t \neq t_i
\]

\[
\Delta x|_{t=t_i} = l_i(x, A(x), B(x), \int_{t-T}^{t} g(s, x(s), A(x(s)), B(x(s))) ds)
\]

\(t \in T, T = \left[\frac{1}{T}, \frac{1}{T} + T\right]
\)

Where \(x \in D \subset R^n, D\) is closed bounded domain.
The vector functions \(f(t, x, y, z, w)\) and \(g(t, x, y, z)\) are defined on the domain

\((t, x, y, z, w) \in R^1 \times D \times D \times D \times D = [-\infty, \infty) \times D \times D \times D \quad \cdots (1.2)
\)

Which are piecewise continuous functions in \(t, x, y, z, w\) and periodic in \(t\) of period \(T\), where \(D_1, D_2\) and \(D_3\) are bounded domain subset of Euclidean spaces \(R^m\).
Let \(l_i(x, y, z, w)\) be continuous vector functions, defined on the domain \((1.2)\) and

\[
l_{i+p}(x, y, z, w) = l_i(x, y, z, w) \quad t_{i+p} = t_i + T \quad \cdots (1.3)
\]

For all \(i \in z^+, x \in D, y \in D_1, z \in D_2, w \in D_3\) and for some number \(p\).
Let the operators \(A\) and \(B\) transform the piecewise continuous functions from the domain \(D\) to the piecewise continuous function in the domain \(D_1\) and \(D_2\) respectively. Moreover \(A(x(t + T)) = A(x(t))\) and \(B(x(t + T)) = B(x(t))\).

Suppose that \(f(t, x, y, z, w), g(t, x, y, z), l_i(t, x, y, z, w)\) and the operators \(A, B\) satisfy the following inequalities:

\[
\|f(t, x, y, z, w)\| \leq M, \|l_i(t, x, y, z, w)\| \leq N, \quad \cdots (1.4)
\]

\[
\|f(t, x_1, y_1, z_1, w_1) - f(t, x_2, y_2, z_2, w_2)\| \leq K(\|x_1 - x_2\| + \|y_1 - y_2\|)
\]

\[
\|g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2)\| \leq Q(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|)
\]

And

\[
\|l_i(t, x_1, y_1, z_1, w_1) - l_i(t, x_2, y_2, z_2, w_2)\| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\| + \|w_1 - w_2\|)
\]

\(i, l \in Z, t \in T, 0 \leq t \leq T\)

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\[\|Ax_{1}(t) - Ax_{2}(t)\| \leq G\|x_{1}(t) - x_{2}(t)\| \]  
\[\|Bx_{1}(t) - Bx_{2}(t)\| \leq H\|x_{1}(t) - x_{2}(t)\| \]  
(1.7)

For all \( t \in R^{1} \), \( x, x_{1}, x_{2} \in D_{1}, y, y_{1}, y_{2} \in D_{2} \), \( z, z_{1}, z_{2} \in D_{2} \) and \( w, w_{1}, w_{2} \in D_{3} \) where \( M, N, K, Q, L \) and \( G, H \) are positive constants.

Consider the matrix

\[ \Lambda = \begin{pmatrix} K_{1} & T \[K_{2} \\
\end{pmatrix} \]  
(1.8)

Where

\[ K_{1} = K[1 + G + H + Q(1 + G + H)], K_{2} = L[1 + G + H + Q(1 + G + H)] \]

We define the non-empty sets as follows

\[ D_{f} = D - M \frac{T}{2}(1 + \frac{4p}{T}), \]  
(1.9)

\[ D_{1f} = D_{1} - GM \frac{T}{2}(1 + \frac{4p}{T}), \]  
(1.9)

and

\[ D_{2f} = D_{2} - HM \frac{T}{2}(1 + \frac{4p}{T}), \]  
(1.10)

\[ D_{3f} = D_{3} - QM \frac{T^{2}}{2}(1 + \frac{4p}{T}). \]  
(1.10)

Furthermore, we suppose that the greatest Eigen-value \( \lambda_{\text{max}} \) of the matrix \( \Lambda \) does not exceed unity, i.e.

\[ \Lambda = K_{1} \frac{T}{2} + pK_{2}(2 + K_{1} \frac{T}{2}) < 1 \]  
(1.11)

**Lemma 2.1.** Let \( f(t) \) be a continuous (piecewise continuous) vector function in the interval \( 0 \leq t \leq T \). Then

\[ \left\| \int_{0}^{t} (f(s) - \frac{1}{T} \int_{0}^{T} f(s) ds) ds \right\| \leq \alpha(t) \max_{t \in [0,T]} \|f(t)\|, \]

Where \( \alpha(t) = 2t(1 - \frac{T}{2}) \). (For the proof see [3]).

Assume that

\[ Ax_{m}(t, x_{0}) = y_{m}(t, x_{0}), \quad Bx_{m}(t, x_{0}) = z_{m}(t, x_{0}) \]  
and

\[ w_{m}(t, x_{0}) = \int_{t-T}^{t} g(s, x_{m}(s, x_{0}), Ax_{m}(s, x_{0}), Bx_{m}(s, x_{0})) ds , \]  

\( m = 0, 1, 2, \ldots \)

**II. APPROXIMATE SOLUTION**

The investigation of a periodic approximate solution of the system (1.1) makes essential use of the statements and estimates given below.

**Theorem 2.1.** If the system of integro-differential equations with impulsive action (1.1) satisfy the inequalities (1.3) to (1.7) and the conditions (1.8), (1.10) has a periodic solution \( x = x(t, x_{0}) \), passing through the point \( (0, x_{0}), x_{0} \in D_{f} \), \( A_{0} \in D_{1f} \) and \( B_{x_{0}} \in D_{2f} \), then the sequence of functions:

\[ x_{m}(t, x_{0}) = x_{0} + \int_{0}^{t} \left[ f(s, x_{m}(s, x_{0}), y_{m}(s, x_{0}), z_{m}(s, x_{0}), w_{m}(s, x_{0})) - \right. \]
\[ \left. \frac{1}{T} \int_{0}^{T} g(s, x_{m}(s, x_{0}), y_{m}(s, x_{0}), z_{m}(s, x_{0}), w_{m}(s, x_{0})) ds \right] ds + \]
\[ + \sum_{0 < t_{i} < T} I_{i}(x_{m}(t_{i}, x_{0}), y_{m}(t_{i}, x_{0}), z_{m}(t_{i}, x_{0}), w_{m}(t_{i}, x_{0})) - \]
\[ - \frac{t}{T} \sum_{i=1}^{p} I_{i}(x_{m}(t_{i}, x_{0}), y_{m}(t_{i}, x_{0}), z_{m}(t_{i}, x_{0}), w_{m}(t_{i}, x_{0})) \]

\( m = 0, 1, 2, \ldots \)

With

\[ x_{0}(t, x_{0}) = x_{0}, \quad m = 0, 1, 2, \ldots \]

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is periodic in $t$ of period $T$, and uniformly convergent as $m \to \infty$ in

$$(t,x_0) \in R^1 \times D_f = (-\infty, \infty) \times D_f$$

To the vector function $x^0(t,x_0)$ defined on the domain (1.2), which is periodic in $t$ of period $T$ and satisfying the system of integral equation

$$x(t,x_0) = x_0 + \int_0^T \{ f(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) -$$

$$- \frac{1}{T} \int_0^T g(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) ds \} ds +$$

$$+ \sum_{0 \leq \iota, \iota} I_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) -$$

$$- \frac{t}{T} \sum_{\iota = 1} T_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0))$$

$$= y + \int_0^T f(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) ds +$$

$$+ \sum_{0 \leq \iota} I_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) -$$

$$- \frac{t}{T} \sum_{\iota = 1} T_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) - x_0 \leq$$

$$\leq (1 - \frac{t}{T}) \int_0^T \{ f(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) \} ds +$$

$$+ \frac{t}{T} \int_0^T \{ f(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) \} ds +$$

$$+ \sum_{0 \leq \iota} I_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) +$$

$$+ \frac{t}{T} \sum_{\iota = 1} T_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) \leq$$

$$\leq \alpha(t) M + 2p M$$

$$\leq M \frac{T}{2} (1 + \frac{4p}{T})$$

From (1.7) and (2.4), we have

$$\|Ax(t,x_0) - Ax_0\| \leq Q M \frac{T}{2} (1 + \frac{4p}{T})$$

$$\|Bx(t,x_0) - Bx_0\| \leq H M \frac{T}{2} (1 + \frac{4p}{T})$$

Which a unique solution of the system (1.1) provided that

$$\|x^0(t,x_0) - x_0\| \leq \frac{MT}{2} (1 + \frac{4p}{T})$$

and

$$\|x^0(t,x_0) - x_m(t,x_0)\| \leq \lambda^m (1 - \lambda)^{-1} M \frac{T}{2} (1 + \frac{4p}{T})$$

for all $m \geq 1$, $t \in R^1$, where the eigen-value $\lambda$ of the matrix $\Delta$ is a positive fraction less than one.

**Proof.** Consider the sequence of functions $x_1(t,x_0), x_2(t,x_0), \cdots, x_m(t,x_0), \cdots$, defined by recurrence relation (2.1), Each of the functions of the sequence is periodic in $t$ of period $T$.

Now, by Lemma 1.1, we have

$$\|x(t,x_0) - x_0\| = || x_0 + \int_0^T \{ f(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) -$$

$$- \frac{1}{T} \int_0^T g(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) ds \} ds +$$

$$+ \sum_{0 \leq \iota} I_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) -$$

$$- \frac{t}{T} \sum_{\iota = 1} T_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) - x_0 \| \leq$$

$$\leq (1 - \frac{t}{T}) \int_0^T \{ f(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) \} ds +$$

$$+ \frac{t}{T} \int_0^T \{ f(s,x(s,x_0), y(s,x_0), z(s,x_0), w(s,x_0)) \} ds +$$

$$+ \sum_{0 \leq \iota} I_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) +$$

$$+ \frac{t}{T} \sum_{\iota = 1} T_i(x(t_\iota,x_0), y(t_\iota,x_0), z(t_\iota,x_0), w(t_\iota,x_0)) \leq$$

$$\leq \alpha(t) M + 2p M$$

$$\leq M \frac{T}{2} (1 + \frac{4p}{T})$$
for all $x_0 \in D_f$. 
Also 
\[
\|w_m(t, x_0) - w_0(t, x_0)\| = \left\| \int_{t-T}^{t} g(s, x_m(s, x_0), Ax_m(s, x_0), Bx_m(s, x_0)) ds - \right. \\
- \left. \int_{t-T}^{t} g(s, x_0, Ax_0, Bx_0) ds \right\| \leq \\
\leq \int_{t-T}^{t} K(||x_m(s, x_0) - x_0|| + ||Ax_m(s, x_0) - Ax_0|| + ||Bx_m(s, x_0) - Bx_0||) ds \\
\leq \int_{t-T}^{t} [K(M \frac{T}{2} (1 + 4p \frac{T}{T})) + GM \frac{T}{2} (1 + 4p \frac{T}{T}) + HM \frac{T}{2} (1 + 4p \frac{T}{T})] ds \\
= KM \frac{T}{2} (1 + 4p \frac{T}{T})(1 + G + H) \quad \cdots (2.8)
\]

For $m = 1$, in (2.1), we get 
\[
\|x_2(t, x_0) - x_1(t, x_0)\| \leq (1 - \frac{T}{T}) \int_{0}^{t} K(||x_1(s, x_0) - x_0|| + ||y_1(s, x_0) - y_0(s, x_0)|| + \|z_1(s, x_0) - z_0(s, x_0)|| + \|w_1(s, x_0) - w_0(s, x_0)|| + \|z_1(t_1, s, x_0) - z_0(t_1, s, x_0)|| + \|w_1(t_1, s, x_0) - w_0(t_1, s, x_0)|| + \|z_1(t_1, t, x_0) - z_0(t_1, t, x_0)|| + \|w_1(t_1, t, x_0) - w_0(t_1, t, x_0)|| + \|z_1(t_1, t, x_0) - z_0(t_1, t, x_0)|| + \|w_1(t_1, t, x_0) - w_0(t_1, t, x_0)|| \\
\leq (1 - \frac{T}{T}) \int_{0}^{t} [K(1 + G + H + Q(L + G + H)) M \frac{T}{2} (1 + 4p \frac{T}{T})] ds + \\
+ \frac{T}{T} \int_{t}^{t} [K(1 + G + H + Q(L + G + H)) M \frac{T}{2} (1 + 4p \frac{T}{T})] ds + \\
+ [L(1 + G + H + Q(L + G + H)) M \frac{T}{2} (1 + 4p \frac{T}{T}) \leq \\
\leq \alpha(t)[K(1 + G + H + Q(L + G + H))] M \frac{T}{2} (1 + 4p \frac{T}{T}) + \\
+ LpMT(1 + G + H + Q(L + G + H))(1 + 4p \frac{T}{T}) \leq \\
\leq N_1 \alpha(t) + M_1 = N_1 \frac{T}{2} + M_1 
\quad \cdots (2.10)
\]

If the following inequality is true 
\[
\|x_m(t, x_0) - x_{m-1}(t, x_0)\| \leq N_{m-1} \alpha(t) + M_{m-1} \\
\leq N_{m-1} \frac{T}{2} + M_{m-1} \quad \cdots (2.11)
\]
for all $m = 1, 2, \cdots$ .
then, we shall to prove that 
\[
\|x_{m+1}(t, x_0) - x_m(t, x_0)\| \leq K_1(N_{m-1} \frac{T}{2} + M_{m-1}) \alpha(t) + 2pK_2(N_{m-1} \frac{T}{2} + M_{m-1}) \\
\quad \cdots (2.12)
\]
for all $m = 0, 1, 2, \cdots$. 

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By mathematical induction, we have
\[ \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \leq N_m \alpha(t) + M_m \leq \frac{T}{2} N_m + M_m, \]  
where
\[ \begin{aligned}
N_{m+1} &= K_1 \frac{T}{2} N_m + K_1 M_m, \\
M_{m+1} &= pK_2 T N_m + 2pK_2 M_m, \quad N_0 = M, \quad M_0 = 2pM.
\end{aligned} \]  
\( m = 0, 1, 2, \ldots \).

It is sufficient to show that all solutions of \((2.14)\) approach zero as \(m \to \infty\), i.e. it is necessary and sufficient that the eigen values of the matrix \(\Lambda\) are assumed to be within a unit circle.

It is well-known that the characteristic equation of the matrix \(\Lambda\) is
\[ K_1 \frac{T}{2} + pK_2 (2 + K_1 \frac{T}{2}) < 1 \]  
\( \cdots (2.15) \)
and this ensures that the sequence of functions \((2.1)\) is convergent uniformly on the domain \((2.2)\) as \(m \to \infty\).

Let
\[ \lim_{m \to \infty} x_m(t,x_0) = x_{\infty}(t,x_0) \]  
\( \cdots (2.13) \)
Since the sequence of functions \((2.2)\) is periodic in \(t\) of period \(T\), then the limiting is also periodic in \(t\) of period \(T\).

Moreover, B lemma 1.1 and \((2.15)\) and the following inequality
\[ \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \leq \sum_{i=0}^{k-1} \|x_{m+i+1}(t,x_0) - x_{m+i}(t,x_0)\| \leq \sum_{i=0}^{k-1} \Lambda^{m+i} M_0 T^2 \frac{1 + 4p}{T} \]
the inequalities \((2.4)\) and \((2.5)\) are satisfied for all \(m \geq 0\).

Finally, we have to show that \(x(t,x_0)\) is unique solution of \((1.1)\). on the contrary, we suppose that there is at least two different solutions \(x(t,x_0)\) and \(r(t,x_0)\) of \((1.1)\).

From \((2.3)\), the following inequality holds:
\[ \|x(t,x_0) - r(t,x_0)\| = h(t), \]  
\( \cdots (2.16) \)
Setting \(\|x(t,x_0) - r(t,x_0)\| = h(t)\), the inequality \((2.16)\) can be written as:
\[ h(t) \leq \left(1 - \frac{t}{T}\right) \int_0^T K_1 s \|x(s,x_0) - r(s,x_0)\| ds + \int_0^T K_1 s \|x(s,x_0) - r(s,x_0)\| ds + \sum_{0 < t_i < t} K_2 \|x(t_i,x_0) - r(t_i,x_0)\| + \sum_{1 \leq t_i < t} K_2 \|x(t_i,x_0) - r(t_i,x_0)\| \]
\[ \cdots (2.17) \]
From \((2.14)\), we have:
\[ \begin{pmatrix} N_{m+1} \\ M_{m+1} \end{pmatrix} = \begin{pmatrix} K_1 T/2 \\ pTK_2 \end{pmatrix} \begin{pmatrix} N_m \\ M_m \end{pmatrix} + \begin{pmatrix} 0 \\ h_m \end{pmatrix} \]  
\( \cdots (2.18) \)
which satisfies the initial conditions \(N_0 = 0, M_0 = h_0\). That is
\[ \begin{pmatrix} N_m \\ M_m \end{pmatrix} = \begin{pmatrix} K_1 T/2 \\ pTK_2 \end{pmatrix} \begin{pmatrix} K_1 \\ 2pK_2 \end{pmatrix}^m \begin{pmatrix} 0 \\ h_m \end{pmatrix} \]  
\( \cdots (2.19) \)
Hence it is clear that if the condition \((2.15)\) is satisfied then \(N_m \to 0\) and \(M_m \to 0\) as \(m \to \infty\).

From the relation \((2.17)\) we get \(h(t) \equiv 0\) or \(x(t,x_0) = r(t,x_0)\), i.e. \(x(t,x_0)\) is a unique solution of \((1.1)\). 

III. **EXISTENCE OF SOLUTION**
The problem of the existence of periodic solution of period $T$ of the system (1.1) is uniquely connected with the existence of zeros of the function $\Delta(x_0)$ which has the form

$$\Delta(x_0) = \frac{1}{T} \int_0^T f(t, x^0(t, x_0), y^0(t, x_0), z^0(t, x_0), w^0(t, x_0)) \, dt +$$

$$+ \sum_{i=1}^p l_i(x^0(t_i, x_0), y^0(t_i, x_0), z^0(t_i, x_0), w^0(t_i, x_0)),$$  \hspace{1cm} \text{(3.1)}

since this function is approximately determined from the sequence of functions

$$\Delta_m(x_0) = \frac{1}{T} \int_0^T f(t, x_m(t, x_0), y_m(t, x_0), z_m(t, x_0), w_m(t, x_0)) \, dt +$$

$$+ \sum_{i=1}^p l_i(x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0))$$ \hspace{1cm} \text{(3.2)}

where $x^0(t, x_0)$ is the limiting of the sequence of functions (2.1). Also

$$y^0(t, x_0) = Ax^0(t, x_0), \quad z^0(t, x_0) = Bx^0(t, x_0) \quad \text{and}$$

$$w^0(t, x_0) = \int_{t-T}^t g(s, x^0(s, x_0), Ax^0(s, x_0), Bx^0(s, x_0)) \, ds.$$ \hspace{1cm} \text{for all} \hspace{1cm} m \geq 1.

Now, we prove the following theorem taking in to ///// that the following inequality will be satisfied for all $m \geq 1.$

$$\|\Delta(x_0) - \Delta_m(x_0)\| \leq \lambda^m (1 - \lambda)^{-1}(K_1 + \frac{p}{T}K_2) \frac{MT}{2} (1 + \frac{4p}{T})$$ \hspace{1cm} \text{(3.3)}

**Theorem 3.1.** If the system of equations (1.1) satisfies the following conditions:

(i) the sequence of functions (3.2) has an isolated singular point $x_0 = x^0,$ $\Delta_m(x_0) \equiv 0,$ for all $t \in R^1$;

(ii) the index of this point is nonzero;

(iii) there exists a closed convex domain $D_4$ belonging to the domain $D_f$ and possessing a unique singular point $x^0$ such that on it is boundary $\Gamma_{D_4}$ the following inequality holds

$$\inf_{x \in \Gamma_{D_4}} \|\Delta_m(x^0)\| \leq \lambda^m (1 - \lambda)^{-1}(K_1 + \frac{p}{T}K_2) \frac{MT}{2} (1 + \frac{4p}{T})$$

for all $m \geq 1.$ Then the system (1.1) has a periodic solution $x = x(t, x_0)$ for which $x(0) \in D_4.$

**Proof.** By using the inequality (3.3) we can prove the theorem is a similar way to that of theorem 2.1.2 [2].

**Remark 3.1.** When $R^n = R^1,$ i.e. when $x^0$ is a scalar, theorem 3.1 can be proved by the following.

**Theorem 3.2.** Let the functions $f(t, x, y, z, w)$ and $l_i(x, y, z, w)$ of system (1.1) be defined on the interval $[a, b]$ in $R^1.$ Then the function (3.2) satisfies the inequalities:

$$\min_{a + \frac{MT}{2} (1 + \frac{4p}{T})} \Delta_m(x^0) \leq - \lambda^m (1 - \lambda)^{-1}(K_1 + \frac{p}{T}K_2),$$

$$\max_{a + \frac{MT}{2} (1 + \frac{4p}{T})} \Delta_m(x^0) \geq \lambda^m (1 - \lambda)^{-1}(K_1 + \frac{p}{T}K_2)$$ \hspace{1cm} \text{(3.4)}

Then (1.1) has a periodic solution in $t$ of period $T$ for which

$$x^0(0) \in \left[a + \frac{MT}{2} (1 + \frac{4p}{T}), b - \frac{MT}{2} (1 + \frac{4p}{T})\right].$$

**Proof.** Let $x_1$ and $x_2$ be any points of the interval

$$[a + \frac{MT}{2} (1 + \frac{4p}{T}), b - \frac{MT}{2} (1 + \frac{4p}{T})]$$

such that

$$\Delta_m(x_1) = \min_{a + \frac{MT}{2} (1 + \frac{4p}{T})} \Delta_m(x^0),$$

$$\Delta_m(x_2) = \max_{a + \frac{MT}{2} (1 + \frac{4p}{T})} \Delta_m(x^0).$$ \hspace{1cm} \text{(3.5)}

By using the inequalities (3.3) and (3.4), we have

$$\Delta_m(x_1) = \Delta_m(x_1) + (\Delta_m(x_1) - \Delta_m(x_2)) < 0,$$

$$\Delta_m(x_2) = \Delta_m(x_2) + (\Delta_m(x_2) - \Delta_m(x_1)) > 0.$$ \hspace{1cm} \text{(3.6)}

The continuity of $\Delta(x^0)$ and by using (3.6), there exists a point $x^0, x^0 \in [x_1, x_2]$, such that $\Delta(x^0) \equiv 0$, i.e. $x = x(t, x_0)$ is a periodic solution in $t$ of period $T$ for which $x^0(0) \in [a + \frac{M_T}{2} (1 + \frac{4p}{T}), b - \frac{M_T}{2} (1 + \frac{4p}{T})]$. 

Similar results can be obtained for other class of integro-differential equations of operators with impulsive action.

In particular, the system of integro-differential equations which has the form

$$
\frac{dx}{dt} = f(t, x, Ax, Bx, \int_{a(t)}^{t} g(s, x(s), Ax(s), Bx(s))ds, \quad t \neq t_i
$$

$$
\Delta x = \sum_{t_i} l_i(x, Ax, Bx, \int_{a(t)}^{t} g(s, x(s), Ax(s), Bx(s))ds, \quad t_i
$$

(3.7)

In this system (3.7), let the vector functions $f(t, x, y, z, w), g(t, x, y, z)$ and scalar functions $a(t), b(t)$ are periodic in $t$ of period $T$, defined and continuous on the domain $(t, x, y, z, w, h) \in R^1 \times G \times G_1 \times G_2 \times G_3 \times G_4 = (-\infty, \infty) \times G \times G_1 \times G_2 \times G_3 \times G_4 \ldots (3.8)$

Let $I_i(x, y, z, w, h)$ be a continuous vector function which are defined on the domain (3.8). A matrix $F(t, s)$ is defined and continuous in $R^1 \times R^1$ and satisfies the condition $F(t + T, s + T) = F(t, s)$ which $\| F(t, s) \| \leq \delta e^{-y(t-s)}$, $0 \leq s \leq t \leq T$, where $\delta, y$ are positive constants. Suppose that the functions $f(t, x, y, z, w, h)$ and $g(t, x, y, z, w, h)$ are satisfying the following inequalities:

$$
\| f(t, x, y, z, w, h) \| \leq M, \| g(t, x, y, z, w, h) \| \leq N, \ldots (3.9)
$$

$$
\| f(t, x_1, y_1, z_1, w_1, h_1) \| - f(t, x_2, y_2, z_2, w_2, h_2) \| \leq K^\ast(\| x_1 - x_2 \| + \| y_1 - y_2 \| + \| z_1 - z_2 \| + \| w_1 - w_2 \| + \| h_1 - h_2 \|);
$$

$$
\| g(t, x_1, y_1, z_1) \| - g(t, x_2, y_2, z_2) \| \leq Q^\ast(\| x_1 - x_2 \| + \| y_1 - y_2 \| + \| z_1 - z_2 \|);
$$

$$
\| I_i(x_1, y_1, z_1, w_1, h_1) - I_i(x_2, y_2, z_2, w_2, h_2) \| \leq L^\ast(\| x_1 - x_2 \| + \| y_1 - y_2 \| + \| z_1 - z_2 \| + \| w_1 - w_2 \| + \| h_1 - h_2 \|);
$$

(3.10)

(3.11)

and

$$
\| Ax_1(t) - Ax_2(t) \| \leq G^\ast(\| x_1(t) - x_2(t) \|),
\| Bx_1(t) - Bx_2(t) \| \leq H^\ast(\| x_1(t) - x_2(t) \|).
$$

(3.12)

for all $t \in R^1, x_1, x_2 \in G, y_1, y_2 \in G_1, z_1, z_2 \in G_2, w_1, w_2 \in G_3, h_1, h_2 \in G_4$, Provided that

$$
Ax(t + T) = Ax(t), Bx(t + T) = Bx(t), I_{i+p}(x, y, z, w, h) = I_i(x, y, z, w, h) \text{ and } t_{i+p} = t_i + T.
$$

Define a non-empty sets as follows:

$$
G_f = G - \frac{M_T}{2} (1 + \frac{4p}{T}), G_i = G_i - G^\ast \frac{M_T}{2} (1 + \frac{4p}{T}),
$$

$$
G_{2f} = G_3 - H^\ast \frac{M_T}{2} (1 + \frac{4p}{T}), G_{4f} = G_4 - \beta Q^\ast \frac{M_T}{2} (1 + \frac{4p}{T}),
$$

$$
G_{5f} = G_5 - \frac{\delta}{y} Q^\ast \frac{M_T}{2} (1 + \frac{4p}{T}).
$$

(3.13)

Furthermore, the largest Eigen-value of $\gamma_{\max}$ of the matrix

$$
\Gamma = \begin{pmatrix}
K_1 & T \\
pK_2 & 2pK_2
\end{pmatrix}
$$

is less than unity, i.e.
\[
\frac{1}{2} \left[ K_1 \left( \frac{T}{3} + 2pK_2 + \frac{K_1 T}{3} + 2pK_2 \right)^2 + \frac{4pK_1 K_2 T}{3} \right] < 1 , \quad \cdots (3.14)
\]
where \( K_1 = K^* \left[ 1 + G^* + H^* + \frac{\delta}{\gamma} Q^* + Q^* (1 + G^* + \beta) \right], \)
\( K_2 = L^* \left[ 1 + G^* + H^* + \frac{\delta}{\gamma} Q^* (1 + G^* + \beta) \right] \) and \( \beta = \max_{0 \leq t \leq T} |b(t) - a(t)|. \)

**Theorem 3.2.** If the system of integro-differential equations with impulsive action (3.7) satisfying the above assumptions and conditions has a periodic solution \( x = \psi(t, x_0) \), then there exists a unique solution which is the limit function of a uniformly convergent sequence which has the form
\[
x_m(t, x_0) = x_0 + \int_0^t \left[ f(s, x_m(s, x_0), y_m(s, x_0), z_m(s, x_0), w_m(s, x_0), h_m(x_0)) - \right.
\]
\[- \frac{1}{T} \int_0^T f(s, x_m(s, x_0), y_m(s, x_0), z_m(s, x_0), w_m(s, x_0), h_m(s, x_0)) ds \] \( + \sum_{0 < t_i < t} I_i (x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0), h_m(t_i, x_0)) \)
\[- \frac{t}{T} \sum_{i=1}^P I_i (x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0), h_m(t_i, x_0)) \] \( \cdots (3.15) \)

with
\[ x_0(t, x_0) = x_0, \quad m = 0, 1, 2, \ldots. \]
The proof is a similar to that of theorem 1.1.
If we consider the following mapping
\[
\Delta_m(x_0) = \frac{1}{T} \left[ \int_0^T f(t, x_m(t, x_0), y_m(t, x_0), z_m(t, x_0), w_m(t, x_0), h_m(t, x_0)) dt + \right.
\sum_{i=1}^P I_i (x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0), w_m(t_i, x_0), h_m(t_i, x_0)) \].
\( \cdots (3.16) \)

Then we can state a theorem similar to theorem 2.2, provided that
\[
\gamma_{\text{max}} = \frac{1}{2} \left[ K_1 \left( \frac{T}{3} + 2pK_2 + \frac{K_1 T}{3} + 2pK_2 \right)^2 + \frac{4pK_1 K_2 T}{3} \right] < 1 , \]
**Remark 3.2.** It is clear that when we put \( I_i \equiv 0 \), we get a periodic solution for the systems (1.1) and (3.7) without introducing impulsive action.

**REFERENCES**