On the Zeros of Analytic Functions inside the Unit Disk

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Abstract: In this paper we find an upper bound for the number of zeros of an analytic function inside the unit disk by restricting the coefficients of the function to certain conditions.

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I. Introduction and Statement of Results

A well-known result due to Enestrom and Kakeya [5] states that a polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

of degree n with

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$$

has all its zeros in $$|z| \leq 1$$.

Q. G. Mohammad [6] initiated the problem of finding an upper bound for the number of zeros of $$P(z)$$ satisfying the above conditions in $$|z| \leq \frac{1}{2}$$. Many generalizations and refinements were later given by researchers on the bounds for the number of zeros of $$P(z)$$ in $$|z| \leq \delta, 0 < \delta < 1$$ (for reference see [1],[2], [4] etc.).

In this paper we consider the same problem for analytic functions and prove the following results:

Theorem 1: Let $$f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0$$ be analytic for $$|z| \leq 1$$, where $$a_j = \alpha_j + i\beta_j, j = 0, 1, \ldots, n$$. If for some $$\rho \geq 0$$,

$$\rho + \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \ldots,$$

then the number of zeros of $$f(z)$$ in $$\frac{|\alpha_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1$$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2 \rho + |\alpha_0| + \alpha_0 + 2 \sum_{j=1}^{\infty} |\beta_j|}{|\alpha_0|}.$$ 

where

$$M = 2 \rho + \alpha_0 + |\beta_0| + 2 \sum_{j=1}^{\infty} |\beta_j|$$

Taking $$\rho = 0$$, the following result immediately follows from Theorem 1:

Corollary 1: Let $$f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0$$ be analytic for $$|z| \leq 1$$, where $$a_j = \alpha_j + i\beta_j, j = 0, 1, \ldots, n$$. If

$$\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \ldots,$$

then the number of zeros of $$f(z)$$ in $$\frac{|\alpha_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1$$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2 \rho + |\alpha_0| + \alpha_0 + 2 \sum_{j=1}^{\infty} |\beta_j|}{|\alpha_0|}.$$ 

where

$$M = 2 \rho + \alpha_0 + |\beta_0| + 2 \sum_{j=1}^{\infty} |\beta_j|$$
\[
\frac{1}{\log \frac{1}{\delta}} \cdot \log \frac{|\alpha_0| + \alpha_0 + 2 \sum_{j=0}^{\infty} |\beta_j|}{|a_0|},
\]

where
\[
M = \alpha_0 + |\beta_0| + 2 \sum_{j=1}^{\infty} |\beta_j|.
\]

If the coefficients \(a_j\) are real i.e. \(\beta_j = 0, \forall j = 0,1,\ldots, n\), we get the following result from Theorem 1:

**Corollary 2**: Let \(f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0\) be analytic for \(|z| \leq 1\), where
\[
\rho + a_0 \geq a_1 \geq a_2 \geq \ldots,
\]
then the number of zeros of \(f(z)\) in \(|a_0| \leq |z| \leq \delta, 0 < \delta < 1\), does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \cdot \log \frac{2 \rho + |a_0| + a_0}{|a_0|},
\]

where
\[
M = 2 \rho + a_0.
\]

Taking \(\rho = (k-1)a_0, k \geq 1\), we get the following result from Theorem 1:

**Corollary 3**: Let \(f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0\) be analytic for \(|z| \leq 1\), where
\[
a_j = \alpha_j + i\beta_j, j = 0,1,\ldots, n.\]
If for some \(k \geq 1\),
\[
k\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \ldots,
\]
then the number of zeros of \(f(z)\) in \(|a_0| \leq |z| \leq \delta, 0 < \delta < 1\), does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \cdot \log \frac{(2k-1)a_0 + |a_0| + 2 \sum_{j=1}^{\infty} |\beta_j|}{|a_0|},
\]

where
\[
M = (2k-1)a_0 + |\beta_0| + 2 \sum_{j=1}^{\infty} |\beta_j|.
\]

Applying Theorem 1 to the function \(-if(z)\), we get the following result:

**Theorem 2**: Let \(f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0\) be analytic for \(|z| \leq 1\), where \(a_j = \alpha_j + i\beta_j, j = 0,1,\ldots, n.\) If for some \(\rho \geq 0\),
\[
\rho + \beta_0 \geq \beta_1 \geq \beta_2 \geq \ldots,
\]
then the number of zeros of \(f(z)\) in \(|a_0| \leq |z| \leq \delta, 0 < \delta < 1\), does not exceed
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$$\frac{1}{\log \frac{1}{\delta}} \log \left( \frac{2\rho + |\beta| + \beta_0 + 2\sum_{j=0}^{\infty} |\alpha_j|}{|\alpha_0|} \right).$$

where

$$M = 2\rho + |\beta_0| + |\alpha_0| + 2\sum_{j=1}^{\infty} |\alpha_j|.$$ 

**Theorem 3**: Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0 \) be analytic for \(|z| \leq 1\). If for some \( \rho \geq 0, \)

$$|\rho + a_0| \geq |a_1| \geq |a_2| \geq \ldots,$$

and for real some \( \beta, \)

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, \ldots, n,$$

then the number of zeros of \( f(z) \) in \(|z| \leq \delta, 0 < \delta < 1\), does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \left( (\rho + |a_0|)(\cos \alpha + \sin \alpha + 1) + \sin \alpha \sum_{j=1}^{\infty} |a_j| \right).$$

where

$$M = \rho + (\rho + |\alpha_0|)(\cos \alpha + \sin \alpha) + \sin \alpha \sum_{j=1}^{\infty} |a_j|.$$ 

Taking \( \rho = 0 \), Theorem 3 reduces to the following result:

**Corollary 4**: Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0 \) be analytic for \(|z| \leq 1\). If,

$$|a_0| \geq |a_1| \geq |a_2| \geq \ldots,$$

and for some real \( \beta, \)

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, \ldots, n,$$

then the number of zeros of \( f(z) \) in \(|z| \leq \delta, 0 < \delta < 1\), does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \left( |a_0|(\cos \alpha + \sin \alpha + 1) + \sin \alpha \sum_{j=1}^{\infty} |a_j| \right).$$

where

$$M = |a_0|(\cos \alpha + \sin \alpha) + \sin \alpha \sum_{j=1}^{\infty} |a_j|.$$ 

Taking \( \rho = (k - 1)|a_0|, \ k \geq 1 \), we get the following result from Theorem 3:
Corollary 5: Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0 \) be analytic for \( |z| \leq 1 \). If,
\[
k|a_0| \geq |a_1| \geq |a_2| \geq \ldots.
\]
and for some real \( \beta \),
\[
|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, \ldots, n,
\]
then the number of zeros of \( f(z) \) in \( \frac{|a_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1 \) does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \log \left( \frac{M}{|f(0)|} \right)
\]
where
\[
M = 2k|a_0|(|\cos \alpha + \sin \alpha| - |a_0| + \sin \alpha \sum_{j=1}^{\infty} |a_j|).
\]

2. Lemmas

For the proofs of the above results we need the following results:

Lemma 1: Let \( f(z) \) be analytic for \( |z| \leq 1 \), \( f(0) \neq 0 \) and \( |f(z)| \leq M \) for \( |z| \leq 1 \).

Then the number of zeros of \( f(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \), does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|f(0)|} \quad \text{(see [7], page 171.)}
\]

Lemma 2: If for some \( \alpha \geq 0 \), \( |a_0| \geq |a_{j-1}| \) and \( |\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots \), for some real \( \beta \), then
\[
|a_j - a_{j-1}| \leq (|a_j| - |a_{j-1}|) \cos \alpha + (|a_j| + |a_{j-1}|) \sin \alpha.
\]

The proof of Lemma 2 follows from a lemma of Govil and Rahman [3].

3. Proofs of Theorems:

Proof of Theorem 1: Consider the function
\[
F(z) = (1 - z) f(z)
\]
\[
= (1 - z)(a_0 + a_1 z + a_2 z^2 + \ldots) = a_0 - (a_0 - a_1)z - (a_1 - a_2)z^2 + \ldots \]
\[
= a_0 + \rho z + (\rho + a_0 - a_1)z - (a_1 - a_2)z^2 - \ldots
+ i\beta_0 - i((\beta_0 - \beta_1)z + (\beta_1 - \beta_2)z + \ldots).
\]

For \( |z| \leq 1 \),
\[
|F(z)| \leq \rho + |a_0| + \rho + a_0 - a_1 + a_1 - a_2 + a_2 - a_3 + \ldots + |\beta_0|
+ |\beta_1| + |\beta_1| + |\beta_2| + \ldots
= 2\rho + |a_0| + a_0 + 2\sum_{j=0}^{\infty} |\beta_j|.
\]
Since $F(z)$ is analytic for $|z| \leq 1$, $F(0) = a_0 \neq 0$, it follows, by using Lemma 1, that the number of zeros of $F(z)$ in $|z| \leq \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \left( 2\rho + |a_0| + a_0 + 2\sum_{j=0}^{\infty} |\beta_j| \right).$$

On the other hand, consider

$$F(z) = (1-z)f(z) = (1-z)(a_0 + a_1z + a_2z^2 + .......) = a_0 - (a_0 - a_1)z - (a_1 - a_2)z^2 + ...... = a_0 + q(z),$$

where

$$q(z) = -(a_0 - a_1)z - (a_1 - a_2)z^2 + ...... = r - (\rho + a_0 - a_1)z - (\alpha_1 - \alpha_2)z^2 - ...... - i\{(\beta_0 - \beta_1)z + (\beta_1 - \beta_2)z^2 + ......\}.$$ 

For $|z| = 1$,

$$|q(z)| \leq \rho + \rho + \alpha_0 - \alpha_1 + \alpha_1 - \alpha_2 + ...... + |\beta_0| + |\beta_1| + |\beta_2| + ...... = 2\rho + a_0 + |\beta_0| + 2\sum_{j=1}^{\infty} |\beta_j| = M.$$ 

Since $q(z)$ is analytic for $|z| \leq 1$, $q(0) = 0$, it follows, by Schwarz’s lemma, that

$$|q(z)| \leq M|z|$$ for $|z| \leq 1$.

Hence for $|z| \leq 1$,

$$|F(z)| = |a_0 + q(z)| \geq |a_0| - |q(z)| \geq |a_0| - M|z| > 0$$ if

$$|z| < \frac{|a_0|}{M}.$$ 

This shows that all the zeros of $F(z)$ lie in $|z| \geq \frac{|a_0|}{M}$. Since the zeros of $f(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $f(z)$ lie in $|z| \geq \frac{|a_0|}{M}$. Thus, the number of zeros of $f(z)$ in

$$\frac{|a_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1$$ does not exceed
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\[ \frac{1}{\log \frac{1}{\delta}} \frac{2 \rho + |\alpha_0| + \alpha_0 + 2 \sum_{j=0}^{\infty} |\beta_j|}{|\alpha_0|} . \]

**Proof of Theorem 2:** Consider the function
\[ F(z) = (1 - z)f(z) \]
\[ = (1 - z)(a_0 + a_1z + a_2z^2 + ...) \]
\[ = a_0 - (a_0 - a_1)z - (a_1 - a_2)z^2 + ... \]
\[ = a_0 + \rho \xi - (\rho + a_0 - a_1)z - (a_1 - a_2)z^2 + ... \]

For \(|z| \leq 1\), we have, by using the hypothesis and Lemma 2,
\[ |F(z)| \leq \rho + |a_0| + \left(\rho \cos (|a_1| + |a_2|) \sin \alpha \right) + \left(\rho \cos (|a_1| + |a_2|) \sin \alpha \right) + ... \]
\[ \leq (\rho + |a_0|)(\cos \alpha + \sin \alpha + 1) + \sin \alpha \sum_{j=1}^{\infty} |a_j| . \]

Since \(F(z)\) is analytic for \(|z| \leq 1\), \(F(0) = a_0 \neq 0\), it follows, by using the lemma 1, that the number of zeros of \(F(z)\) in \([0, \delta, 0 < \delta < 1]\), does not exceed
\[ \frac{1}{\log \frac{1}{\delta}} \frac{(\rho + |a_0|)(\cos \alpha + \sin \alpha + 1) + \sin \alpha \sum_{j=1}^{\infty} |a_j|}{|a_0|} . \]

Again, consider the function
\[ F(z) = (1 - z)f(z) \]
\[ = (1 - z)(a_0 + a_1z + a_2z^2 + ...) \]
\[ = a_0 - (a_0 - a_1)z - (a_1 - a_2)z^2 + ... \]
\[ = a_0 + \rho \xi - (\rho + a_0 - a_1)z - (a_1 - a_2)z^2 + ... \]

where
\[ q(z) = -(a_0 - a_1)z - (a_1 - a_2)z^2 + ... \]
\[ = \rho \xi - (\rho + a_0 - a_1)z - (a_1 - a_2)z^2 + ... \]

For \(|z| = 1\), by using lemma 2, we have,
\[ |q(z)| \leq \rho + (\rho + |a_1| + |a_2|) \cos \alpha + (\rho + |a_1| + |a_2|) \sin \alpha \]
\[ + (\rho + |a_1| + |a_2|) \cos \alpha + (\rho + |a_1| + |a_2|) \sin \alpha + ... \]
\[ \leq \rho + (\rho + |a_0|)(\cos \alpha + \sin \alpha) + \sin \alpha \sum_{j=1}^{\infty} |a_j| = M . \]

Since \(q(z)\) is analytic for \(|z| \leq 1\), \(q(0) = 0\), it follows, by Schwarz’s lemma, that
\[ |q(z)| \leq M|z| \text{ for } |z| \leq 1. \]

Hence for \(|z| \leq 1\),
\[ |F(z)| = |a_0 + q(z)| \]
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\[ |a_0| - |q(z)| \geq |a_0| - M|z| > 0 \]

if

\[ |z| < \frac{|a_0|}{M}. \]

This shows that all the zeros of \( F(z) \) lie in \( |z| \geq \frac{|a_0|}{M} \). Since the zeros of \( f(z) \) are also the zeros of \( F(z) \), it follows that all the zeros of \( f(z) \) lie in \( |z| \geq \frac{|a_0|}{M} \). Thus, the number of zeros of \( f(z) \) in \( \frac{|a_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1 \), does not exceed

\[
\frac{1}{\log \frac{1}{\delta}} \log \left( (\rho + |a_0|)(\cos \alpha + \sin \alpha + 1) + \sin \alpha \sum_{j=1}^{\infty} |a_j| \right). 
\]

References