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Riesz Fractional Derivatives in Fractional Dimensional Space

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Abstract *The Fourier transform method is used to solve fractional Poission's equation with Riesz fractional derivative of order α. It is shown that the solution is given in terms of the fractional dimensional space D. Gauss law for the electrostatic problem is given and the total electric flux is obtained in terms of α and D.*

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I. Introduction

The history of fractional calculas [1-4], dating back to the 17^{th} century, is almost as long as that of the integer-order calculus. Mandelbrot [5] proposed that there is a lot of fractional dimension in nature and there is a close connection between fractional Brownian motion and Riemann-Liouville fractional calculus. From then on, the fractional calculus has been used successfully to study man complex system. The problem of electromagnetic theory was investigated by English [6] and the solution of Poission's equations are given for the cases of fractional charge distributions in integer space. Very recently Muslih and Baleanu [7] have studied the fractional multipole expansion in fractional dimensional space using the fractional dimensional Laplacian defined by Stillinger [8].

Although the embedding space in our world is three dimension (3D) Euclidean space, the motion of material objects is not always in three dimensions. The dimensionality depends on restrain condition [p-11].

Fractional dimensional space has successfully been used as an effective physical descriptions of confinement in low-dimensional systems. First applied by He [9-11], this approach replaces the real confining structure with an effective space, where the measure of the anisotropy or confinement is given by the non-integer dimension.

Many of the investigations into low-dimensional semiconductors structures have used a mathematical basis introduced by Stillinger [7] in which he described integration on a space of α dimensions and provided a generalization of the Laplace operator on this space. Recent progress includes the description of a single coordinate momentum operator in this fractional dimensional space based on generalized Wigner relation [12,13] presenting realization of parastatics [14]. In some applications, the fractional dimensions appears as an explicit parameter when the physical problem is formulated in α dimensions in such a way that α may be extended to non-integer values, as in Wilson's study of quantum filed theory models in less than four dimension[15], or in the approach t quantum mechanics by Stillinger [7]. It is worth while to mention that the experimental measurement of the dimensional α of our real worled is give by $\alpha = (3 \pm 10^{-6})$ [17,15]. The fractional value of α a agrees with the experimental physical observations that in general relativity, gravitational fields ae understood to be geometric perturbations (curvatures) in our space-time [16], rather than entities residing within a flat space-time. Beside, Zeilinger and Svozil [17] noted that the current discrepancy between theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of space α is $\alpha = 3 - (5.3 \pm 2.5) \times 10^{-7}$.

The formalism from [7] has been applied to problem such as exactions [1], 18-24, magnetoxcitions [25] , impurities [26], polarons [27], and superconductivity [28], often successfully mirroring computational results in specified problems. Also , the putative fractional dimension may be viewed as an effective dimension of compactified higher dimensions or as a manifestation of a non-microscopic lattice structure of space [29].

In this paper we solve Poission's equation using the Riesz fractional derivative [2,4,30,31] and its Fourier transformation. Also we work out the fractional multipoles in fractional dimensional space and finding a compact expression using the definition of Gegenbauer polynomials.

II. Fractional Poission's Equation

One of the effective methods to find the solution of the electrostatic problems is to find the solutions of Poission's equation and then one can find the electric field for the Source Charge. The starting point is to consider the fractional Possion's equation as follows.

$$(-\Delta)^{\alpha/2}\phi(\mathrm{r}) = \frac{\rho}{\epsilon_0}, \qquad 1 < \alpha \le 2, \tag{1}$$

Where ρ is the source electric charge density, $\vec{r}$ is the D dimensional vector $\Delta = \frac{\partial^2}{\partial \mathrm{r}^2}$ is the Laplician and the operator $(-\Delta)^{\alpha/2}$ is the α dimensional generalization of fractional quantum Riesz derivative [2,4,30,31]

$$(-\Delta)^{\alpha/2}\psi(\mathrm{r},\mathrm{t}) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} e^{\mathrm{ik.r}|\mathrm{K}|^{\alpha} d^D k} \int_{R^D} \psi(\mathrm{r},\mathrm{t}) e^{-ik.r} d^D r. \tag{2}$$

To solve (1) , we use the Fourier transforms as

$$F\phi(\mathrm{r}) = g(k) = \int_{R^D} \phi(\mathrm{r}) e^{-ik.r} d^D r. \tag{3}$$

$$F\rho(\mathrm{r}) = f(K) = \int_{R^D} \rho(\mathrm{r}) e^{ik.r} d^D r. \tag{4}$$

The inverse Fourier transform reads as

$$F^{-1}\phi(\mathrm{r}) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} g(\mathrm{k}) e^{-ik.r} d^D k. \tag{5}$$

$$F^{-1}\rho(\mathrm{r}) = \left(\frac{1}{2\pi}\right)^D \int_{K^D} f(\mathrm{k}) e^{-ik.r} d^D k. \tag{6}$$

Applying the Fourier transform with respect to spatial variable *x* we have

$$g^{(k)} = \frac{f(k)}{k^{\alpha}} \tag{7}$$

Hence, we obtain the solutions ϕ (r) as

$$\phi(\mathrm{r}) = \int_{R^D} G(\mathrm{r}-\mathrm{r}') \frac{\rho(\mathrm{r}')}{\epsilon_0} d^D \mathrm{r}'. \tag{8}$$

Where, G (r-r') is the Green's function (*kernel*) and is given by

$$G(\mathrm{r}-\mathrm{r}') = \left(\frac{1}{2\pi}\right)^D \int_{K^D} \frac{e^{\mathrm{ik(r-r')}}}{k^{\alpha}} d^D \mathrm{k}. \tag{9}$$

Now using the following transformation defined in [2,4]

$$\int_{K^D} \varphi(\mathrm{k}) e^{\mathrm{ik(r-r')}} d^D k = \int_0^{\alpha} \varphi(\rho)\rho^{D-1} d\rho \int_{s_{D-1}} e^{\mathrm{i}\rho(\mathrm{r}-\mathrm{r}')-\alpha} d\sigma \tag{10}$$

We obtain

$$\int_{K^D} \frac{e^{ik.(r-r')}}{k^{\alpha}} d^D k = \int_0^{\alpha} \rho^{D-\alpha-1} d\rho \int_{s_{D-1}} e^{i\rho(r-r')-\alpha} d\sigma \tag{11}$$

Nothing that

$$\int_{s_{D-1}} f(r-\sigma)\, d\sigma = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \int_{-1}^{1} f(|x|t)(1-t^2)^{(D-3)/2} dt \tag{12}$$

And also

$$J_{\nu}(x) = \frac{(x/2)^{\nu}}{\Gamma(\nu+1/2\sqrt{\pi}} \int_{-1}^{1} e^{ixt}(1-t^2)^{(\nu-1/2)} dt \tag{13}$$

Where $J_{v(x)}$ is the Bessel function of the first kind . we arrive at

$$\int_{s_{D-1}} e^{ix-\alpha} d\sigma = \frac{2\pi^{D/2}}{|x|^{D/2-1}} J_{d/2-1}(|x|). \tag{14}$$

The Green's Function (9) takes the form

$$G(r-r') = (2\pi)^{-D/2} \int_o^{\alpha} \frac{\rho^{D-\alpha-1} J_{D/2}(\rho\,|r-r'|)}{(\rho\,|r-r'|)^{D/2-1}} d\rho \tag{15}$$

Using the Mellin transform of the Bessel function [2,32]

$$\int_o^{\alpha} \rho^{\beta} J_{\upsilon}(\rho) dp = \frac{2^{\beta}\Gamma\left(\frac{\upsilon+\beta+}{2}\right)}{\Gamma\left(\frac{\beta-\upsilon+1}{2}\right)} \tag{16}$$

We obtain

$$G(r-r') = \frac{1}{2^{\alpha}\pi^{D/2}} \frac{\Gamma\left(\frac{D-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{|r-r'|^{D-\alpha}} \tag{17}$$

The solution of poission's equation (1) is given by

$$\phi(r)_{\alpha,D} = k_{\alpha,D}\; a \int \frac{\rho(r')d^D r'}{|r-r'|^{D-\alpha}} \tag{18}$$

Where the constant k_{α}, D is defined as

$$k_{\alpha.D} = \frac{1}{2^{\alpha}\pi^{D/2\in 0}} \frac{\Gamma\left(\frac{D-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \tag{19}$$

For α =2 and D =3, We have k2,3 = $\frac{1}{4\pi\in 0}$

For different value of α and *D*, we obtain the potential for any *D* fractional dimensional space and for any order *Riesz* fractional derivative of order α via the relation

$$(-\Delta)^{\alpha/2}\left(\frac{1}{|\mathrm{r}-\mathrm{r}'|^{D-\alpha}}\right)=\frac{2^{\alpha}\pi^{D/2}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{D-\alpha}{2}\right)}\delta^{D}(\mathrm{r}-\mathrm{r}') \tag{20}$$

Where δ^{D} (r-r') is the *D* dimensional fractional Dirac delta function . For α =2 and *D*=3 we have.

$$\Delta-\left(\frac{1}{|\mathrm{r}-\mathrm{r}'|}\right)=-4\pi\delta^{3}(\mathrm{r}-\mathrm{r}') \tag{21}$$

III. Gauss's Law

Gauss's Law gives the relation between electric field and the charge enclosed in a closed Gauss surface. To derive Gauss's law in *D* dimension fractional space, let us consider a closed *D* dimensional sphere of radius *R* with its center at the origin of the coordinate system. The total flux of the electric field on the surface of the closed sphere is

$$\oint \overline{\mathrm{E}}.\overline{d}A=\int_0^{\pi}\left(-\frac{\partial\phi(r)}{\partial r}\right)r^{D-1}(\sin\theta)^{D-2}d\theta=\frac{qr^{\alpha-2}}{2^{\alpha-1}\in_0}\frac{\Gamma\left(\frac{D-\alpha}{2}\right)(D-2)}{\Gamma\left(\frac{\alpha}{2}\right)} \tag{22}$$

For α = 2 and *D* = 3

$$\oint \vec{E}.\vec{d}A=\frac{q}{\in_o} \tag{23}$$

IV. Fractional Multipole Expansion of Riesz Potential of Order α in Fractional *D* Dimensional Space

Multipole, expansion of sources is a very well known subject in electromagnetism, and has been studied extensively (e.g.[133-35] In this section we will obtain the mulipole expansion in fractional space . The potential (18) can be expanded using the definition of generating function of Gegnbauer polynomials and considering that *r* > *r*'

$$\frac{1}{|\mathrm{r}-\mathrm{r}'|^{D-\alpha}}=\sum_{l-0}^{\infty}\frac{c_1^{(D-\alpha)/2}(\cos\vartheta)r^{rl}}{r^{D-\alpha+1}}.\ r>r' \tag{24}$$

Where $\hat{r}.\hat{r}'=\cos\vartheta$ and $C_l^{(D-\alpha)/2}(\cos\vartheta)$ are the Gegenbauer polynomials in cos ϑ and the forms of the first few Gegenbauer polynomials are given by

$$C_0^{(D-\alpha)/2}(x)=1 \tag{25}$$

$$C_1^{(D-\alpha)/2}(x)=(D-\alpha)x, \tag{26}$$

$$C_2^{(D-\alpha)/2}(x)a=\left(\frac{D-\alpha}{2}\right)((D-\alpha+2)x^2-1) \tag{27}$$

The potential (18), may be expressed as

$$\phi(\mathrm{r})_{\alpha.D}=k_{\alpha,D}\sum_{i=0}^{\infty}\frac{q_1^{D.\alpha}}{r^{D-\alpha+1}}, \tag{28}$$

Where $q_l^{D,\alpha}$ are the fractional multipole terms of order *i* and are given by

$$q_1^{D,\alpha} = \int \rho(\mathrm{r}')r^{rl}C_1^{(D-\alpha)/2}(\cos\vartheta)\,d^Dr'. \qquad (29)$$

For *l*=0 we have only fractional monopole term as

$$\phi(\mathrm{r})_{\alpha.D} = k_{\alpha.D}\frac{q}{r^{D-\alpha}} \qquad (30)$$

Where *q* represent the monopole charge.

V. Conclusion

In this paper we have given an application of fractional calculus in electromagnetic theory and introduce a solution of the fractional Poission's equation with Riesz derivative . The Fourier transform method is used to solve this equation and it is observed that fractional derivative and the fractional dimensional space are connected simultaneously via relation (20), which means that fractionality in the derivatives is due toe the fractionality in the space. Another interesting result we have obtained is the new definition of the constant k_α, D

The importance of this study is that, it is considered as as intermediate cases in electromagnetic theory, where one can study the fractional multipole and the fractional space (see for example (24)). Besise it will be the starting point for given the solution of fractional Helmoltz equation $(-\Delta^{\alpha/2}\phi(\mathrm{r},t)+(-\frac{\partial 2}{\partial t^2})^{\beta/2}\phi(\mathrm{r},t) = f(\mathrm{r},t),1,<\alpha,\beta \le 2$. in *D* dimensional fractional space and this topic is now under consideration of the authors.

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