International Journal of Engineering Inventions
e-ISSN: 2278-7461, p-ISSN: 2319-6491
Volume 12, Issue 11 [November. 2023] PP: 18-23

A Mathematical Analysis in Double - Diffusive Convection Coupled With Cross – Diffusions In Completely Confined Fluids

Hari Mohan*
Department of Mathematics, ICDEOL, Himachal Pradesh University, Summer Hill Shimla-5, India.
** E-mail:hmmathhpu@gmail.com*

Abstract
Some energy relationships and limitations for the growth rate of a disturbance in the problems of double-diffusive convection coupled with cross – diffusions of Veronis' and Stern's type configurations for Rivlin - Ericksen viscoelastic fluids completely confined in an arbitrary region in the three dimensional Euclidean space R^3 *are derived in the present paper..For Veronis configuration, the total kinetic energy associated with a neutral or unstable disturbance is shown to be greater than its total concentration energy in the parameter regime* $\frac{R_S'\sigma\, l^4}{\tau^2 \lambda_0^2 k_2^2} \le (1-F)$*, and the principal of exchange of stabilities is valid. Further, the complex growth rate* p *of an arbitrary perturbation, neutral or unstable, for Veronis type configuration is shown to lie inside a semi-circle whose centre is at* $\left(-\frac{\tau\lambda_0 k_2}{l^2}, 0\right)$ *and (radius)*^{2} = $\frac{R_S'\sigma}{(1-F)}$*.Similar results also shown to follow for the Stern's type configuration.*
Keywords: *Double-diffusive convection; Rivlin-Ericksen viscoelastic fluid; Rayleigh numbers; Dufour number; Soret number; Prandtl number.*

Date of Submission: 01-11-2023 Date of acceptance: 11-11-2023

I. Introduction

The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour effects [1,2] . An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. Although it is clear that the thermosolutal and Soret-Dufour problems are quite closely related, their relationship has never been carefully elucidated. They are in fact, formally identical and identification is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. The analysis of double diffusive convection becomes complicated in case when the diffusivity of one property is much greater than the other. Further, when two transport processes take place simultaneously, they interfere with each other and produce cross diffusion effect. The Soret and Dufour coefficients describe the flux of mass caused by temperature gradient and the flux of heat caused by concentration gradient respectively. The coupling of the fluxes of the stratifying agents is a prevalent feature in multicomponent fluid systems. In general, the stability of such systems are also affected by the cross-diffusion terms. Generally, it is assumed that the effect of cross diffusions on the stability criteria is negligible. However, there are liquid mixtures for which cross diffusions are of the same order of magnitude as the diffusivities. There are only few studies available on the effect of cross diffusion on double diffusion convection largely because of the complexity in determining these coefficients. Hurle and Jakeman [3] have studied the effect of Soret coefficient on the double–diffusive convection problem. They have reported that the magnitude and sign of the Soret coefficient were changed by varying the composition of the mixture. McDougall [4] has made an in depth study of double diffusive convection, where in both Soret and Dufour effects are important.

In all the above studies, the fluid has been considered to be Newtonian. However, with the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. The Rivlin-Ericksen [5] fluid is such fluid. Many research workers have paid their attention towards the study of Rivlin-Ericksen fluid. Srivastava and Singh [6] have studied the unsteady flow of a dusty elastico-viscous Rivlin-Ericksen fluid through channel of different cross-sections in the presence of the time dependent pressure gradient. Sharma and Kumar [7] have studied the thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid acted on by a uniform rotation and found that rotation has a stabilizing effect and introduces oscillatory modes in the system. Sharma and Kumar [8] have studied the thermal instability in Rivlin-Ericksen elastico-viscous fluid in hydromagnetics.

In Banerjee et.al [9, 10], an attempt has been to establish the relationships between various energies in magnetothermohaline convection of Veronis [11] and Stern [12] types. The analysis made brings out that the total kinetic energy associated with a disturbance is greater than sum of its total magnetic and concentration energies in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{R_S\sigma}{\tau^2\pi^2}\le 1,$ for Veronis' configuration, whereas for Stern's configuration the total kinetic energy associated with a disturbance is greater than sum of its total magnetic and thermal energies in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{|R_T|\sigma}{\pi^2}\le 1,$ where $Q,\sigma,\sigma_1,R_S,\tau$ *and* R_T respectively represent the Chandrasekhar number, the thermal Prandtl number, the magnetic Prandtl number, the concentration Rayliegh number, the Lewis number and the thermal Rayliegh number.

The aim of the present paper is to extend, and to show that these results for double-diffusive convection coupled with cross diffusion in viscoelastic fluids completely confined in an arbitrary region in the three dimensional Euclidean space R^3,
in the absence of magnetic field ($Q=0$) are of wider generality and applicability than the simple context of the horizontal layer geometry for which they have been derived without considering the non Newtonian fluid and cross-diffusions effects.

II. Mathematical Formulation and Analysis

The relevant governing non-dimensional linearized perturbation equations in the present case with time dependence of the form exp (pt) ($p=p_r+ip_i$) are given by:

$$\left(1-F\right)\frac{p}{\sigma}\vec{q}=-\nabla\left(P\right)-curl\ curl\ \vec{q}+R_T\theta\,\hat{k}-R_S\phi\,\hat{k} \quad (1)$$

$$\left(\nabla^2-p\right)\theta+D_T\nabla^2\phi=-\vec{q}\cdot\hat{k}, \quad (2)$$

$$\left(\tau\nabla^2-p\right)\phi+S_T\nabla^2\theta=-\vec{q}\cdot\hat{k}, \quad (3)$$

and $$\nabla\cdot\vec{q}=0. \quad (4)$$

In the above Eqs. $\vec{q}(x,y,z)$, $P(x,y,z)$, $\theta(x,y,z)$, *and* $\phi(x,y,z)$ respectively denote the perturbed velocity, pressure, temperature, concentration and are complex valued functions defined on V, F== $\frac{\upsilon'}{d^2}$ is the viscoelastic parameter, $R_T=\frac{g\alpha\beta d^4}{\kappa\upsilon}$ is the thermal Rayleigh number, $R_S=\frac{g\alpha'\beta' d^4}{\kappa'\upsilon}$ is the concentration Rayleigh number, $\sigma=\frac{\upsilon}{\kappa}$ is the Prandtl number, $\tau=\frac{\eta_1}{\kappa}$ is the Lewis number, $D_T=\frac{\beta_2 D_f}{\beta_1\kappa}$ is the Dufour number, $S_T=\frac{\beta_1 S_f}{\beta_2\eta}$ is the Soret number, and $\hat{k}$ is a unit vertical vector. Further, with d as the characteristic length, the equations have been cast into dimensionless forms by using the scale factors $\frac{\kappa}{d},\ \frac{d^2}{\kappa},\ \beta d,\ \frac{\rho\upsilon\kappa}{d^2},$ *and* $\beta' d$ for velocity, time, temperature, pressure, concentration respectively.

We seek solutions of Eqs. (1)- (4) in the simply connected subset V of R^3 subject to the following boundary conditions:

$\vec{q}=0=\theta=\phi$ *on* S (Rigid bounding surface with fixed temperature and mass concentration) (5)

We now prove the following theorems:

Theorem 1. If $(p,\vec{q},\theta,\phi)$, $p = p_r + ip_i$, $p_r \geq 0$, 0 < F <1, is a solution of Eqs.(1)-(5) with $R'_T > 0$, $R'_S > 0$ and if $\frac{R'_T \sigma\, l^4}{\tau^2 \lambda_0{}^2 k_2^2} \leq (1-F)$, then

$$\int_V |\vec{q}|^2 dv > \frac{R'_S \sigma}{(1-F)} \int_V |\phi|^2\, dv \qquad \text{and} \qquad p_i = 0,$$

where l is the smallest distance between two parallel planes that just contains V and λ_0 (>2) is a constant.

Proof. We introduce the transformations

$$\breve{\vec{q}} = (S_T + B)\,\vec{q}\,,\ \ \tilde{\theta} = E\theta + F\phi\,,$$

$$\tilde{\phi} = S_T\theta + B\phi \quad (6)$$

where

$$\text{B} = -\frac{1}{\tau}A, \quad \text{E} = \frac{S_T + B}{D_T + A}A, \quad \text{F} = \frac{S_T + B}{D_T + A}D_T$$

and A is a positive root of the equation

$$A^2 + (\tau - 1)A - \tau\, S_T D_T = 0\,.$$

The systems of Eqs. (1)- (5), upon using the transformations (6) assume the following forms:

$$(1-F)\frac{p}{\sigma}\vec{q} = -\nabla P - curl\ curl\ \vec{q} + R'_T\theta\,\hat{k} - R'_S\varphi\,\hat{k}\,, \tag{7}$$

$$\left(\ k_1\nabla^2 - p\ \right)\theta = -\vec{q}\cdot\hat{k}\,, \tag{8}$$

$$\left(\ k_2\tau\nabla^2 - p\ \right)\phi = -\vec{q}\cdot\hat{k}\,, \tag{9}$$

$$\nabla\cdot\vec{q} = 0 \tag{10}$$

with

$$\vec{q} = 0 = \theta = \phi\ on\ \ S \tag{11}$$

where

$k_1 = 1 + \frac{\tau\, D_T S_T}{A}$, $k_2 = 1 - \frac{S_T D_T}{A}$ are positive constants

and $R_T^{/} = \frac{(D_T + A)(R_T B + R_S S_T)}{BA - S_T D_T}$, $R_S^{/} = \frac{(S_T + B)(R_S A + R_T D_T)}{BA - S_T D_T}$

are respectively the modified thermal Rayleigh number and the modified concentration Rayleigh number.
The sign tilde has been omitted for simplicity.

Using Gauss divergence theorem, boundary condition (11) and the solenoidal character of velocity field, it follows that

$$\int_V \vec{q}^{\,*}\cdot\nabla(P)dv = 0\,, \tag{12}$$

$$\int_V \vec{\text{q}}^{\,*}\cdot\text{curl curl}\,\vec{\text{q}}\text{dv} = \int_V |\text{curl }\vec{\text{q}}|^2\text{dv}\,, \tag{13}$$

$$\int_V \theta^*\nabla^2\theta dv = -\int_V |\nabla\theta|^2\, dv = \int \theta^*\nabla\theta^*\, dv\,, \tag{14}$$

and $$\int_V \phi^*\nabla^2\phi dv = -\int |\nabla\phi|^2\, dv = \int \phi^*\nabla\phi^*\, dv\,, \tag{15}$$

where * indicates complex conjugation.

Further, multiplying Eq. (9) by its complex conjugate, integrating over V and using integral relation (15), we have

$$k_2^2\tau^2\int_V\left|\nabla^2\phi\right|^2 dv+2p_r\tau\, k_2\int_V\left|\nabla\phi\right|^2 dv+\left|p\right|^2\int_V\left|\phi\right|^2 dv=\int_V\left|\vec{q}.\hat{k}\right|^2 dv\,. \tag{16}$$

Now since $\phi=0$ on S, it therefore follows by Joseph [13] that,

$$\int_V\left(\nabla\phi.\nabla\phi^*\right)dv\ge\frac{\lambda_0}{l^2}\int_V\left|\phi\right|^2 dv\,, \tag{17}$$

and

$$\int_V\left|\nabla\phi\right|^2 dv\le\int_V\left|\phi\right|^2\left|\nabla^2\phi\right|^2 dv\le\left\{\int_V\left|\phi\right|^2 dv\right\}^{\frac{1}{2}}\left\{\int_V\left|\nabla^2\phi\right|^2 dv\right\}^{\frac{1}{2}},$$

(Schwartz' inequality)

which upon utilizing (17) gives

$$\int_V\left|\nabla^2\phi\right|^2 dv\ge\frac{\lambda_0^2}{l^4}\int_V\left|\phi\right|^2 dv\,. \tag{18}$$

Eq. (16) together with inequality (18) implies that

$$R_S'\sigma\int_V\left|\phi\right|^2 dv-\left(1-F\right)\int_V\left|\vec{q}\right|^2 dv<\left\{\frac{R_S'\sigma\, l^4}{k_2^2\tau^2\lambda_0^2}-\left(1-F\right)\right\}\int_V\left|\vec{q}\right|^2 dv\,.$$

Consequently, if $\frac{R_S'\sigma\, l^4}{\tau^2\lambda_0{}^2k_2^2}\le\left(1-F\right)$, then we must have

$$\int_V\left|\vec{q}\right|^2 dv\;>\;\frac{R_S'\sigma}{\left(1-F\right)}\int_V\left|\phi\right|^2 dv\;. \tag{19}$$

Forming dot product of Eq. (7) with $\vec{q}\,*$, integrating over the domainV, and making use of Eqs. (8) and (9) appropriately, we get

$$\begin{aligned}&\left(1-F\right)\frac{p}{\sigma}\int_V\left(\vec{q}.\vec{q}\,*\right)dv+\int_V\left|\nabla\vec{q}\right|^2 dv+R_T'\int_V\left(k_1\left|\nabla\theta\right|^2+p\left|\theta\right|^2\right)dv\\&\qquad-R_S'\int_V\left(k_2\tau\left|\nabla\varphi\right|^2+p\left|\varphi\right|^2\right)dv-R_T'I_1+R_S'I_2=0,\end{aligned} \tag{20}$$

where

$$I_1=2\,\mathrm{Re}\left\{\int_V\left(\;\theta.\vec{k}\;\right).\vec{q}*dv\right\} \tag{21}$$

$$I_2=2\,\mathrm{Re}\left\{\int_V\left(\;\phi.\vec{k}\;\right).\vec{q}*dv\right\}, \tag{22}$$

and Re denotes the real parts.

Equating imaginary parts of both sides of Eq. (20) and taking $p_i\ne 0$, we have

$$\left(1-F\right)\int_V\left|\vec{q}\right|^2 dv-R_S'\sigma\int_V\left|\phi\right|^2 dv+R_T'\sigma\int_V\left|\theta\right|^2 dv=0 \tag{23}$$

Eq. (23) obviously cannot hold in view if inequality (19).Hence we must have

$$p_i=0. \tag{24}$$

This completes the proof of the theorem.

Theorem 1, from the physical point of view implies that for the problem of double-diffusive convection coupled with cross- diffusion in viscoelastic fluids completely confined in an arbitrary region in the three dimensional Euclidean space R^3 for the Veronis type, the total kinetic energy associated with a neutral or unstable disturbance is greater than its total concentration energy in the parameter regime $\frac{R'_S\sigma\, l^4}{\tau^2\lambda_0{}^2k_2^2} \le (1-F)$, and the principal of exchange of stabilities is valid.

Theorem 2. If $(p,\vec{q},\theta,\phi)$, $p = p_r + ip_i$, $p_r \ge 0$, 0<F<1 is a solution of Eqs. (7) - (11) with $R'_T > 0$, $R'_S > 0$, then

$$\left\{p_r + \left(\frac{\tau\lambda_0 k_2}{l^2}\right)\right\}^2 + p_i{}^2 < \frac{R'_S\sigma}{(1-F)}.$$

Proof. It follows from Eqs. (16) and (23), upon using inequalities (17) and (18) that

$$\frac{\left(\tau^2\lambda_0{}^2k_2{}^2\right)}{l^4} + 2p_r\left(\frac{\tau\lambda_0 k_2}{l^2}\right) + |p|^2 < \frac{R'_S\sigma}{(1-F)},$$

which can be written as

$$\left\{p_r + \left(\frac{\tau\lambda_0 k_2}{l^2}\right)\right\}^2 + p_i{}^2 < \frac{R'_S\sigma}{(1-F)}. \qquad (25)$$

This completes the proof of the theorem.

Theorem 2 implies that the complex growth rate $p(= p_r + p_i)$ of an arbitrary $(p_i \ne 0)$ perturbation neutral $(p_r = 0)$ or unstable $(p_r > 0)$, for the double-diffusive convection coupled with cross-diffusion in completely confined viscoelastic fluids for Veronis configuration must lie inside a semi-circle whose centre is at $\left(-\frac{\tau\lambda_0 k_2}{l^2}, 0\right)$ and (radius)^2 = $\frac{R'_S\sigma}{(1-F)}$ in the three dimensional Euclidean space R^3

Theorem 3. If $(p,\vec{q},\theta,\phi)$, $p = p_r + ip_i$, $p_r \ge 0$ is a solution of Eqs. (7) - (11) with $R'_T < 0$, $R'_S < 0$ and if $\frac{|R'_T|\sigma\, l^4}{\lambda_0{}^2k_1{}^2} \le (1-F)$, then

$$\int_V |\vec{q}|^2 dv > \frac{|R'_T|\sigma}{(1-F)}\int_V |\theta|^2 dv \quad \text{and} \quad p_i = 0. \qquad (26)$$

Proof. Follows similarly as in Theorem 1.

The essential contents of Theorem 3 are similar to those of Theorem 1.

Theorem 4. If $(p,\vec{q},\theta,\phi)$, $p = p_r + ip_i$, $p_r \ge 0$ is a solution of Eqs. (7)-(11) with $R'_T < 0$, $R'_S < 0$, then

$$\left\{p_r + \left(\frac{\lambda_0 k_1}{l^2}\right)\right\}^2 + p_i{}^2 < \frac{|R'_T|\sigma}{(1-F)}. \qquad (27)$$

Proof. Follows similarly as in Theorem 2.

The essential contents of Theorem 4 are similar to those of Theorem 2.

References

[1]. D.D. Fitts, Nonequlibrium Thermodynamics (McGraw Hill, New York, 1962).
[2]. S.R. De Groot and P Mazur, Nonequilibrium Thermodynamics (North Holland, Amsterdam, 1962).
[3]. D.T.J Hurle.and E. Jakeman., J. Fluid Mech., **47**, 668 (1971).
[4]. T.J. McDougall., J. Fluid Mech., **126**, 379 (1983).
[5]. R.S. Rivlin and J.L. Ericksen,. J. Rat. Mech. Anal., **4,** 323 **(**1955).
[6]. R.K.Srivastava and K.K. Singh., Cal. Math. Soc.**,** **80**, 286 (1988).
[7]. R.C. Sharma. and P. Kumar, Z. Naturforsch., **51a**, 821 (1996).
[8]. R.C. Sharma and P. Kumar., Z. Naturforsch., **52a**, 369 (1997).
[9]. M.B. Banerjee, J.R. Gupta. and S.P. Katyal, Indian J. Pure Appl. Math., **18,** 856 (1987).

[10]. M.B. Banerjee, J.R. Gupta. and S.P. Katyal, J. Math. Anal. Appl., **144**, 141 (1989).
[11]. G. Veronis, J. Mar. Res., **23**, 1 (1965).
[12]. M.E. Stern, Tellus, **12**, 172 (1960).
[13]. D.D. Joseph, Stability of Fluid Motions, Vol. I (Springer Verlag, NY, 1976).