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Tripartitions of Natural Numbers

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Abstract
A tripartition of a natural number, n, is an expression of the form n = a + b + c. It is known that for n > 17, n has a tripartition such that a < b < c, a > 1, and a, b, and c are pairwise relatively prime. Various results concerning these tripartitions and several variations are presented.

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I. Partitions

A *partition* of a natural number, n, is an expression of the form $n = n_1 + n_2 + \ldots + n_r$, where the n_i's are natural numbers, and r is the number of summands of the partition. See [1]. Partitions have excited number theorists since the time of the great 18th century Swiss mathematician, Euler. In determining the number of partitions of n, we must distinguish between *ordered* and *unordered* partitions. In the later case, we treat, say, 9 = 1 + 5 + 3 and 9 = 3 + 1 + 5 as the same partition. The eight ordered partitions of 4 are:

4	2 + 2	1 + 1 + 2
1 + 3	2 + 1 + 1	1 + 1 + 1 + 1
3 + 1	1 + 2 + 1	

The five unordered partitions of 4 are:

4 1 + 3 2 + 2 1 + 1 + 2 1 + 1 + 1 + 1

Here a few facts about partitions.

1. The number of ordered partitions of n is 2^{n-1}.
2. The number of partitions of n, where each summand is odd equals the number of partitions of n, where the summands are distinct.
3. Given the natural number $k \le n$, the number of partitions of n, where each summand is less than or equal to k equals the number of partitions of n, where there are at most k summands.

II. Tripartitions

In this article, we shall write partitions in increasing order. That is, $n_1 \le n_2 \le \ldots \le n_r$. We shall confine ourselves to *tripartitions*, that is, partitions for which $r = 3$. It is known [2, 3] that for $n > 17$, we can write $n = a + b + c$, where

1. $a < b < c$
2. $a > 1$, and
3. a, b, and c are pairwise relatively prime (abbreviated PRP).

That is, $n > 17$ has a pairwise relatively prime tripartition.

Lemma 1: (1) If n is odd, then each of a, b, and c must be odd. **(2)** If n is even, exactly one of a, b, and c must be even.

Proof: (1) $a + b + c$ is odd if either they are all odd, or two of them are even. As the second possibility is incompatible with the PRP condition, the result follows. **(2)** $a + b + c$ is even if either they are all even, or one of them is even. As the first possibility is incompatible with the PRP condition, we are done. ■

Remark 1: Here are solutions for $n < 17$:

10 = 2 + 3 + 5	14 = 2 + 5 + 7	16 = 2 + 5 + 9
12 = 2 + 3 + 7	15 = 3 + 5 + 7	

The reader may wish to prove that there are no solutions for n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 17.

Remark 2: $b \ge a + 1$ and $c \ge a + 2$, so $n \ge a + (a + 1) + (a + 2) = 3a + 3$. Then $3a + 3 \le n$, so

$$\boxed{a \le \frac{n-3}{3}}$$

yielding an upper bound for a.

III. Consecutive and Arithmetic Tripartitions

A *consecutive* tripartition of n may be written $n = a + (a + 1) + (a + 2)$. The PRP condition implies that a, and therefore, $a + 2$, are odd. Letting $a = 2k + 1$, we have

$$n = 2k + 1 + (2k + 2) + (2k + 3) = 6k + 6 = 6(k + 1)$$

so $n = 0 \pmod 6$ is a necessary condition. To see that it is sufficient, $n = 6m = 2m + 2m + 2m = (2m - 1) + 2m + (2m + 1)$, which clearly satisfies the PRP condition since

$$\text{GCD}[(2m - 1), 2m] = \text{GCD}[2m, (2m + 1)] = 1$$

This follows since the GCD of two consecutive numbers is 1. Finally, $2m - 1$ and $2m + 1$ are odd and two apart, so $\text{GCD}[(2m - 1), (2m + 1)] = 1$

Three numbers, x, y, and z form an *arithmetic progression* if there exists a number, d, such that $y = x + d$ and $z = y + d$. Observe that y is the arithmetic mean of x and z. Also, the three numbers can be written $y - d$, y, $y + d$. For our purposes, we seek *arithmetic tripartitions* of n, which requires $y - d$, y, $y + d$ to have the PRP condition. In particular, $y - d$ must be odd. Otherwise, $y + d = (y - d) + 2d$ would also be even. Examine the following *arithmetic tripartitions* of $n = 42$ that are PRP:

13 + 14 + 15
11 + 14 + 17
9 + 14 + 19
5 + 14 + 23
3 + 14 + 25

We skipped 7 + 14 + 21, as this arithmetic tripartition is not PRP. (In fact, 7 divides each summand!) Now examine the *arithmetic tripartitions* of $n = 60$ that are PRP:

19 + 20 + 21
17 + 20 + 23
13 + 20 + 27
11 + 20 + 29
9 + 20 + 31
7 + 20 + 33
3 + 20 + 37

Once again, we skipped 15 + 20 + 25 and 5 + 20 + 35 (5 divides each summand). The eligible values of a must be relatively prime to n. As n goes to infinity, the set of natural numbers relatively prime to it also goes to infinity, so the number of arithmetic partitions of n that are PRP approaches infinity.

IV. Tripartitions with Given Differences, $r = b - a$ and $s = c - b$

Given natural numbers, n, r, and s, does there exist a PRP tripartition $n = a + b + c$, such that $b - a = r$ and $c - b = s$? If so, how many such tripartitions does n have? Equivalently, are there values of k such that $n = (k - r) + k + (k + s)$? If so, $n = 3k + s - r$, implying the necessary condition, $n = s - r \pmod 3$. Then we have

$$\boxed{k = \frac{n-(s-r)}{3}}$$

. For example, given $n = 23$, $r = 4$, and $s = 6$, we find that $k = 7$, so $23 = 3 + 7 + 13$. If $n = 35$, $r = 4$, and $s = 6$, then $k = 11$, so $35 = 7 + 11 + 17$. These two tripartitions are PRP. On the other hand, if $n = 26$, $r = 4$, and $s = 6$, then $k = 8$, so $26 = 4 + 8 + 14$ which has the given differences, 4 and 6, but isn't PRP. Lemma 2, below, will be useful in establishing which of partitions are PRP.

Lemma 2: If $\text{GCD}[k, t] = 1$, then $\text{GCD}[k, k \pm t] = 1$.

Proof: We will prove the *contrapositive* by contradiction. Assume that $\text{GCD}[k, k \pm t] = d > 1$. Then $d \mid k$ and $d \mid (k \pm t)$. Then $k = ad$ and $k \pm t = bd$, that is, $ad \pm t = bd$. So $\pm t = bd - ad$, implying that $d \mid k$. It follows that $\text{GCD}[k, t] = d > 1$, a contradiction. ■

Using Lemma 2, if $\text{GCD}[k, r] = \text{GCD}[k, s] = 1$, then $\text{GCD}[k, k - r] = \text{GCD}[k, k + s] = 1$. It remains to determine additional conditions to ensure that $\text{GCD}[k - r, k + s] = 1$.

V. Removing the Restriction $a \neq 1$

Lemma 3: Every natural number, $n \geq 7$, satisfies $n = b + c$ where $1 < b < c$ and $\text{GCD}[b, c] = 1$. **Proof:** If n is odd, let $n = 2k + 1$. Then $n = k + (k + 1)$. Thus $7 = 3 + 4$, $9 = 4 + 5$, etc. If n is even, let k satisfy $1 < k < n - 1$ and $\text{GCD}[n, k] = 1$. It follows that $n - k$ satisfies these two conditions, so $n = k + (n - k)$. ■

Theorem 1: Let $n \geq 8$. Then $n = a + b + c$, where

1. $a < b < c$
2. $a \geq 1$, and
3. a, b, and c are PRP

Proof: Given $n \geq 8$, we have by Lemma 3, $n - 1 = b + c$ where $1 < b < c$ and $\text{GCD}[b, c] = 1$. Then $n = 1 + (n - 1) = 1 + b + c$. ■

VI. Primitive Tripartitions

The set, $S = \{n_1, n_2, \ldots, n_r\}$, where $r \geq 2$ and no member divides any other, is called *primitive*. Clearly, 1 cannot belong to S. If 2 belongs to S, then all other members must be odd. We assume, WLOG, that $n_1 < n_2 < \ldots < n_r$. Note that a primitive set need not have the PRP property. The sets, $\{k + 1, k + 2, k + 3, \ldots, 2k\}$, of length, k, form an infinite class of primitive sets for $k \geq 2$. The primes form a primitive set of infinite cardinality.

A partition of n is called *primitive*, if its summands form a primitive set. If there are exactly three summands, we have a *primitive tripartition*. For example, 17 = 4 + 6 + 7 is a primitive partition. It is not PRP. Clearly, PRP tripartitions of n are primitive tripartitions, whereas the converse of generally false.

Example 1: The primitive partitions of $n = 19$ are

3 + 4 + 5 + 7	8 + 11	
4 + 6 + 9	7 + 12	4 + 15
3 + 5 + 11	6 + 13	3 + 16
9 + 10	5 + 14	2 + 17

The primitive tripartitions of 19 are in bold font. Note that 4 + 6 + 9 does not have the PRP property.

Example 2: Let $n = (k + 1) + (k + 2) + (k + 3) + \ldots + (k + k) = k^2 + (1 + 2 + 3 + \ldots + k) = k^2 + \frac{k(k+1)}{2} = \frac{2k^2 + k(k+1)}{2} = \frac{3k^2 + k}{2}$. Then $n = \frac{3k^2 + k}{2}$ has a consecutive primitive tripartition of length k. We see from this example that there exist arbitrarily lengthy primitive partitions.

1, 2, 3, 4, and 6 are the only natural numbers that do not have primitive partitions. If $p \geq 5$ is prime, it has $\left\lfloor \frac{p}{2} \right\rfloor - 1$ primitive *bipartitions*. (It may also have longer primitive partitions.) See Example 1.

Lemma 4: Let n be an odd natural number with the primitive partition, $n = a + b + c$, where $a < b < c$. Then $a \geq 3$.

Proof: We know that $a \neq 1$. If $a = 2$, then b and c must be odd, implying that n is even, which contradicts the hypothesis that n is odd. ■

Let $g(n)$ denote the number of primitive partitions of n, and let $h(n)$ denote the maximum length among the primitive partitions of n.

Open Questions: Determine $g(n)$ and $h(n)$ and study their properties.

References

[1]. M.Lewinter, J.Meyer, Elementary Number Theory with Programming, Wiley & Sons. 2015.
[2]. W.Sierpinski, 250 Problems in Number Theory, American Elsevier, New York 1970.
[3]. J.Roberts, Lure of the Integers, Spectrum Series, MAA. 1992