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Construction of Hadamard Matrix by the Tensorial Product Method (Kronecker Product)

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ABSTRACT: *Hadamard matrix are algebraic structures in the sense that they form an important subclass of the class of matrix and must therefore conform to all the algebraic rules obeyed by matrix under the usual operations of addition and multiplication. On the other hand, Hadamard matrix are also combinatorial structures since the elements $+1$ of -1 which the matrix is composed must follow certain patterns. In this paper, we will present a recursive construction method for constructing the Hadamard matrix based on the Kronecker product.*

Keywords: *Matrix, Hadamard matrix, construction, Kronecker product*

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I. INTRODUCTION

A matrix is a concise and useful way to uniquely represent and work with linear transformations. In particular, each linear transformation can be represented by a matrix, and each matrix corresponds to a unique linear transformation. A matrix, and its determinant, are extremely important concepts in mathematics. Matrix is widely used in a variety of fields, including applied mathematics, computer science, economics, engineering, operations research, statistics, and other disciplines.

On the other hand, the Hadamard matrix forms an important class of matrix, with applications ranging from binary codes, and information theory, to quantum mechanics. This class of matrix has the orthogonal property, which was introduced by Sylvester in 1867 [1]. In mathematics, the Hadamard matrix was named after the French mathematician Jacques Hadamard, who was further studied. Hadamard matrix look like simple matrix structures, where they are square matrix with elements $+1$ or -1, and have orthogonal row vectors and orthogonal column vectors. In geometric terms, this means that each pair of rows in the Hadamard matrix is represented by two normal vectors. n-dimensional parallelepiped constructed from the rows of a Hadamard matrix $n \times n$ has the maximum possible n-dimensional volume between parallels constructed from vectors whose elements are limited to absolute value 1. Equivalently, a Hadamard matrix has the maximum determinant among matrix, with entries of absolute value less than or equal to 1, and so the Hadamard matrix is the maximum determinant of the solution of a problem [2]. Hadamard matrix has a simple but elegant structure. They have been studied for over 150 years, yet there is still much to learn about their characteristics, existence, construction, as well as their interaction with other modern fields of mathematics and science in general [3].

Definition1A Hadamard matrix is a H square matrix whose elements must be ± 1 and which satisfies the condition:

$$HH^T = nI_n$$ (**Error! Reference source not found.**.1)

where H^T is the transposed matrix of the matrix H, represents the order of the matrix $H$and I_n is $n \times n$ the unit matrix [4].

Examples of order 1, 2 and 4 are as follows:

$$H_1 = [1],\ H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ H_4 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$ (**Error! Reference source not found.**.2)

We can denote +1 and -1 as and the Hadamard matrix can look like this [5]:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}$$ (**Error! Reference source not found.**)3

Hadamard matrix has a very wide variety of applications in modern technology. A method of construction is of fundamental importance in a concrete application [6]. There are several methods to construct a Hadamard matrix, such as:

- Sylvester method
- Walsh method
- Tensor product method (Kronecker's product)
- Williamson method
- Paley Method

This paper aims to examine the construction of the Hadamard matrix by the method of tensor product or Kronecker product. In mathematics, the Kronecker product, sometimes denoted $\otimes$, is an operation on two matrix of arbitrary size that results in a block matrix. It is a generalization of the outer product (which is denoted by the same symbol) from vectors to matrix and gives the matrix of the linear scheme of the tensor product concerning a standard choice of basis. The Kronecker product should be distinguished from ordinary matrix multiplication, which is a completely different operation [7].

Tensorial product or the Kronecker product for the matrix A and B is denoted by $A^{\otimes} B$ and is defined as follows [8]:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{n1}B & a_{n1}B & \cdots & a_{nn}B \end{bmatrix} = (a_{ij}B)_{n\times n} \quad (1.4)$$

II. METHOD

It is important in application to know when Hadamard matrix exist and to be able to construct them. The simplest way to construct a Hadamard matrix is the Kronecker product of the smallest Hadamard matrix.

There is now a theorem due to James Sylvester (1814–1897) stating that the Kronecker product of two Hadamard matrix is another Hadamard matrix. So, if we can find the Hadamard matrix of order p and q, we can take their Kronecker product to form a Hadamard matrix of order pq. In particular, we can obtain the Kronecker product of the above matrix times itself n to form a Hadamard matrix of order 2^n. It is not known for which there exists a n Hadamard matrix of order n, but the above construction shows that there exists a Hadamard matrix of any order that is a power of 2. If $n > 2$, then a necessary condition for the existence of a matrix of Hadamard 's is that n it is a multiple of 4. It is assumed that even this condition is sufficient. At the time of writing, the smallest multiple of 4 for which no one has found a Hadamard matrix is 668. Note that, for example, there are Hadamard matrix of order 20, but you cannot find a multiplying Hadamard matrix of orders 4 and 5, because there is no Hadamard matrix of order 5 [9].

Definition: Let be $A = (A_{ij}) \in M_{n\times m}(M_l(K))$ and $B = (B_{ij}) \in M_{p\times q}(M_l(K)), np \times mq\ MMs.$

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1m}B \\ A_{21}B & A_{22}B & \dots & A_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nm}B \end{bmatrix} \quad (1.5)$$

Theorem1(Construction of Hadamard matrix using the Kronecker product). If there exists two Hadamard matrix of order m and n for every integer m and n, then there exists an n Hadamard matrix of order mn formed by the "Kronecker product" of the two.

Proof of $m = n = 2$: We will describe how to construct a Hadamard matrix of order 4 from the two Hadamard matrix of order 2 shown earlier.

$$H_4 = H_{2\times 2} = H_2 \otimes H_2 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} \quad (1.6)$$

In general, $H_m \otimes H_n$ it's in whatever shape you want it to be $+1$ in H_n it is replaced by H_m and in whomever you want -1 in H_n it is replaced by $-H_m$. It is an easy check by straightforward computation that the resulting matrix is Hadamard.

- Hadamard matrix of order 2^t for every integer $t = 1, 2, 3, ..$.
- There exists a Hadamard matrix of order 2^n if there exists a Hadamard matrix of order n.

Generalization from No and Song (2002)

- Hadamard matrix of order m and m, all n Hadamard matrix $C_m = (cij)$ of order (B_i are not necessarily distinct), then there exists a Hadamard matrix of order nm of [9] the form $B_1, B_2, \dots, B_m$:

$$\begin{bmatrix} c_{11}B_1 & c_{12}B_2 & \cdots & c_{1m}B_m \\ c_{21}B_1 & c_{22}B_2 & \dots & c_{2m}B_m \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}B_1 & c_{n2}B_2 & \cdots & c_{nm}B_m \end{bmatrix} \tag{1.7}$$

In this section, we present the construction of the Hadamard matrix using the Kronecker product. This construction is recursive and requires at least one Hadamard matrix to use it. Therefore, it is most useful to use this in conjunction with some of the other techniques for constructing the Hadamard matrix [11]. An example of tensorial rolling or Kronecker's product is given below:

$$H_8 = H_2 \otimes H_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

and so on,

$$H_{2^n} = H_2 \otimes H_2 \otimes \cdots \otimes H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1.8}$$

III. CONCLUSIONS

The importance of the Kronecker product stems from the fact that it naturally correlates with different areas of mathematics. The tensor product as observed in this paper plays a very important role in the construction of Hadamard matrix and in other areas of pure and applied mathematics. Some special cases of this application are also considered. Here we presented Kronecker products and the construction of the Hadamard matrix by this method, where we obtained some results that will be useful in several applications. So, by identifying the construction of the Hadamard matrix, we were able to derive results that can be applied in various fields. Like any application of mathematics, the Hadamard matrix can be used in many different fields [12].

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