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Merging Techniques Applied to Difference Necklaces

Tejmal Rathore, *Independent Researcher,*
G-803, Country Park, Dattapada Road, Borivali (E), Mumbai 400066, India

ABSTRACT: *Three merging techniques for deriving the difference necklaces (necklaces which have absolute difference of two adjacent numbers belonging to a set of integer numbers) are presented. There are many possible necklaces for the same number. Some applications are suggested.*

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I. INTRODUCTION

Recently, Rathore [1] has given a trio-based method for generating necklaces of length *N* such that *the sum of the two adjacent numbers is a perfect square.* A *trio* is a set of 3 numbers (a, b, c) such that ($a + b$) and ($b + c$) satisfy a given criterion. The method was extended to the sum a perfect cube [2]. Finally, the method was used for any set of specified numbers [3]. In this paper, we present three merging techniques for deriving the difference necklaces (DNs) (*necklaces with the absolute value of the difference of two adjacent numbers belonging to a set of specified numbers).* In section 2, we derive the condition for the existence of DNs. Section 3 gives the derivations of DNs of higher lengths from that of the lower ones by merging technique. Some applications are suggested in Section 4. Section 5 gives the conclusions.

II. EXISTENCE OF DIFFERENCE NECKLACES

A necklace of length *N* which has all the numbers 1-*N* will be expressed as

$$N;(S = a_1, a_2, \dots , a_n) = p_1,\ p_2, p_3, \dots\dots, p_N \tag{1}$$

where a_is are the permissible set of numbers for the absolute difference of two neighbouring numbers. Without any loss of generality, we assume

$$a_i > a_{i-1}. \tag{2}$$

For a DN,

$$|p_N - p_1| \in S. \tag{3}$$

To begin with, we will restrict to the simplest

$$S = a_1, a_2. \tag{4}$$

Let the highest number *N* be located between p_i and p_j Then

$$p_i, p_j < N. \tag{5}$$

$$\lfloor p_j - p_i \rfloor = (a_1, a_2), \tag{4}$$

$$N - p_i = (a_1, a_2), \tag{5}$$

$$N - p_j = (a_1, a_2). \tag{6}$$

$$p_i = N - (a_1, a_2) < N. \tag{7}$$

$$p_j = N - (a_1, a_2) < N. \tag{8}$$

Lemma:

If $a_1 < a_2 \le N$, then there exists a DN $N;(a_1, a_2)$ such that

N	is any integer	when	$a_1 = 1, a_2 = 2$
	is odd number		$a_1 = 1, a_2 > 2$ even
	is even number		$a_1 = 1, a_2 \ge 3$ odd
	does not exist		Both a_1, a_2 even
	$= a_1 + a_2$		Any odd-even combination of a_1, a_2.

(9)

Proof: From Equations (7), (8)

$$p_i = N - (a_1, a_2) < N. \tag{10}$$

$$p_j = N - (a_1, a_2) < N. \tag{11}$$

These leads to $a_1, a_2 > 0$. Therefore, the valid pair is $(a_1, a_2 > a_1)$.

N may be odd and even, and a_1, a_2 may be both odd, both even, a_1 odd and a_2 even, a_1 even and a_2 odd. Thus, there are 8 combinations to be examined to derive all the possible necklaces. After trying several examples and using the property that the *difference of two numbers x and y is even when both x and y are even or odd, and odd when one is even and the other is odd,* we found the results depicted in Equation (9). This completes the proof.

Note that the cases $a_1 = 1$, and a_2 is either even, or odd > 1 are included in (9) with the restriction $a_1 < a_2 \le N$ whereas the following two lemmas in [4] impose the restriction $N \ge 2a_2$.

Lemma (1): If $a_2 \ge 2$ is even then there exists a necklace N;(1, a_2) for all $N \ge 2a_2$.

Lemma (2): if $a_2 \ge 3$ and odd then there exists a necklace N;(1, a_2) for all even $N \ge 2a_2$.

Thus, these lemmas do not cover cases when $N < 2a_2$, such as 3;(1,2) = 1,3,2; $N < 4$, 7;(1,4) = 1,2,6,7,3,4,5; $N < 8$, 4;(1,3) = 1,4,3,2; $N < 6$.

Examples of all the cases of Equation (9) are given below.

i) a_1=1 and a_2 = 2, N is any integer
   5;(1,2) = 3,5,4,2,1
ii) a_1 = 1, a_2 > 2 and even, N odd
   7;(1,6) = 1,7,6,5,4,3,2
   9;(1,6) = 3,9,8,2,1,7,6,5,4
iii) a_1=1, and $a_2 \ge 3$ and odd, N even
   6;(1,3) = 3,6,5,4,1,2
   8;(1,5) = 7,2,1,6,5,4,3,8
iv) Both a_1 and a_2 even, *no necklace is possible*.
v) a_1+a_2 = N
   4;(1,3) = 1,4,3,2
   7;(1,6) = 2,3,4,5,6,7,1
   7;(2,5) = 1,3,5,7,2,4,6
   7;(3,4) = 3,7,4,1,5,2,6

III. TECHNIQUES FOR DERIVING HIGHER LENGTH NECKLACES

3.1 *Technique 1: By merging a single number*

A) By merging N+1

To derive the necklace of length N+1from that of N, insert N+1 between two consecutive numbers p_i and p_j. Replacing N by N+1 in Equations (10) and (11), we get

$$p_i = (N+1) - (a_1, a_2) < (N+1). \tag{12}$$
$$p_j = (N+1) - (a_1, a_2) < (N+1). \tag{13}$$

Since,

$$p_i = \{(N+1-a_1), (N+1-a_2)\}, \tag{14}$$
$$p_j = \{(N+1-a_1), (N+1-a_2)\}, \tag{15}$$
$$|p_j - p_i| = |\{(N+1-a_1), (N+1-a_2)\} - \{(N+1-a_1), (N+1-a_2)\}|$$
$$\rightarrow |p_j - p_i| = (a_2 - a_1) \tag{16}$$

But, from Equation (6), $|p_j - p_i| = a_1, a_2$. Thus, Equation (16) becomes $(a_2 - a_1) = a_1, a_2$. This will be satisfied only if

$$2a_1 = a_2. \tag{17}$$

Thus, a_2 *must be double of* a_1. Hence from Equation (16),

$$|p_j - p_i| = a_1. \tag{18}$$

The conditions of Equations (17) and (18) should be satisfied in order to get the DN by this technique.

From Equation (18), $|p_N - p_1| = a_1$

In the new necklace, $(N+1) - 1 = a_1 \rightarrow N = a_1$. Thus, *N can be merged in between the last and the first number only when* $N = a_1$.

If we express N;(1,2) as *odd* numbers (in increasing order) and *even* numbers (in decreasing order), then relation is

$$(N+1); (1,2): 1,3,5, \dots, (N+1), \dots, 6,4,2. \tag{19}$$

If we express *N*;(1,2) as *even* numbers (in increasing order) and *odd* Numbers (in decreasing order), then relation is

$$(N+1);(1,2): 2,4,6,\ldots(N+1),\ldots 5,3,1 \tag{20}$$

Procedure
Find all the possible pairs (p_i, p_j) which satisfy Equation (18). Select only those pairs which are present in *N*;(*S*).

Example 1: Let DN be 2;(1,2) = 1,2. In view of Equation (18),

$$|p_i - p_j| = 1.$$

But

$$(p_i, p_j) = (1,2).$$

Merge 3 between 1 and 2 to get 3;(1,2) = 1,3,2. Similarly, 4;(1,2) = 1,3,4,2, and so on.

Example 2: Consider 4;(1,3) = 1,2,3,4. It does not satisfy the condition in Equation (17). Hence, we cannot get the necklaces of higher length by this method.

***B) By merging* 1**

Increase all the numbers in *N* by 1 and then merge 1 between 2 and 3 suitably. 4;(1,2) = 1,3,4,2 → 2,4,5,3. Then 5;(1,2) = 2,4,5,3,1, 6;(1,2) = 2,4,6,5,3,1 and 7;(1,2) = 1,3,5,7,6,4,2.
From the above examples, we note
(i) the DN *N*;(1,2) exist for any length *N*.
(ii) the DN of length *N* (odd) has (*N*-1)/2 odd numbers in decreasing order right of *N* and (*N*+1)/2 even numbers in decreasing order right of *N*.
(iii) DN of length *N* (even) has (*N*+2)/2 odd numbers in decreasing order left of *N* and (*N*-2)/2 even numbers in decreasing order right of *N*.

B) Direct method

With the above observations (ii) and (iii), it is easy to find the DNs of any length *N* directly, rather than starting from the necklace of lower length, from the relation

$$N;(1,2): \begin{cases} N,\ldots,3,1,2,4,\ldots,(N-1), & N \text{ odd} \\ (N-1),\ldots 3,1,2,4,\ldots,N & N \text{ even} \end{cases}. \tag{21}$$

Thus, the recursive relation is

$$(N+2);(1,2): \begin{cases} (N+2), N;(1,2),(N+1) & N \text{ odd} \\ (N+1), N;(1,2),(N+2) & N \text{ even} \end{cases} \tag{22}$$

Examples
N odd

$$15;(1,2): 15,13,11,9,7,5,3,1,2,4,6,8,10,12,14$$
$$17;(1,2): 17,15,13,11,9,7,5,3,1,2,4,6,8,10,12,14,16$$

N even

$$12;(1,2): 11,9,7,5,3,1,2,4,6,8,10,12.$$
$$14;(1,2): 13,11,9,7,5,3,1,2,4,6,8,10,12,14.$$

3.2 *Techniques 2: By merging two numbers*

Let us merge two numbers (*N*+1),(*N*+2) in between p_i and p_j of *N*;(a_1, a_2). *Since we are inserting two numbers, the method will work for necklaces of even (odd) length to even (odd) length.*
Since p_i, p_j belong to *N*;(a_1, a_2), and p_1 is 1,

$$N \geq p_i, p_j \neq 1. \tag{23}$$

Since *N*+1 and *N*+2 are the two consecutive numbers, their difference is 1. Hence, one of a_is must be 1. Number 1 being the lowest number,

$$a_1 = 1. \tag{24}$$

Now

$$\begin{aligned} p_i &= (N+1) - (1, a_2) \leq N \\ &= N, (N+1-a_2). \end{aligned} \tag{25}$$

Let $p_i > p_j$. Then

$$p_j = p_i - (1, a_2) \leq N$$

$$= N - 1, N - a_2, N + 1 - 2a_2 > 0. \qquad (26)$$

However,

$$\begin{aligned} p_j &= (N+2) - (1, a_2) \\ &= (N + 2 - a_2) \leq N \text{ and} > 0 \end{aligned} \qquad (27)$$

Therefore, only the common numbers in Equations (25) and (26) will be the valid p_j.
Similarly, when $p_i < p_j$,

$$p_j = (N + 2 - a_2) \leq N \text{ and} > 0. \qquad (28)$$

However,

$$\begin{aligned} p_j &= (N+2) + (1, a_2), \leq N \\ &= (N + 2 - a_2) \leq N \text{ and} > 0, \end{aligned} \qquad (29)$$

Equations (27) and (29) are the same. Hence only Equations (25), (26) need to be considered. Choose the common numbers from Equations (26) and (27) for $p_i > p_j$, and all the values given by Equation (28) for $p_i < p_j$.

Procedure: Find p_i from Equation (25). Choose the common values given by Equations (26) and (27). Find $(p_i, p_j), p_i > p_j$, and $p_i < p_j$. Choose those values of (p_i, p_j) which are present in N;(1,a_2). Find the necklaces.

Example 3: Let 4;(1,3): 1,2,3,4.
From Equation (25), $p_i = 4,2$. Common value from Equations (25) and (26) is 3, for $p_i > p_j$. 3 is a common number in the two sets. However, p_j = 3 for $p_i < p_j$. Thus, (p_i, p_j) = (4,3) $p_i > p_j$,(2,3) ($p_i < p_j$).
The two necklaces are

6;(1,3)$_1$: 1,2,3,6,5,4 and 6;(1,3)$_2$ =1,2,5,6,3,4

Similarly,

8;(1,3)$_1$: 1,2,3,6,7,8,5,4 and 8;(1,3)$_2$: 1,2,3,6,5,8,7,4 and 8;(1,3)$_3$:1,2,5,8,7,6,3,4

and so on.

We note that the last terms in Equation (25) also contributes when N = 6. Therefore, the number of DNs increases with N = 6. Hence, a greater number of DNs are obtained when N >6. Perhaps, the total number of necklaces is not important, since each one contains the same numbers arranged differently. Hence one is good enough.

In the above examples, we have chosen a necklace which fixes a_is. But in practice, a_is are specified, and we must find a necklace if it exists.

Example 4: Let DN be 5;(1,4): 1,2,3,4,5.
Here a_is are perfect squares 1,4. From Equation (25), $p_i = 5,2$ and from Equations (26) and (27), $p_j = 4$ and 3, respectively. There is no common value for p_j. From Equation (29), p_j = 3. Thus, (p_i,p_j) = (2,3). Therefore, 7;(1,4) = 1,2,6,7,3,4,5.

Example 5: Let 10;(1,4,9) = 1,2,3,4,5,6,7,8,9,10.
Here (a_1, a_2, a_3) = (1,4,9). Because of the additional term a_3, Equations (25), (26), (27), and (29) become, respectively,

$$\begin{gathered} p_i = N, (N + 1 - a_2), (N + 1 - a_3) \\ p_j = N, N - 1, N - a_{2,}, N - a_3, N + 1 - 2a_2, N + 1 - a_2 - a_3, N + 1 - 2a_3. \\ p_j = (N + 2 - a_2), (N + 2 - a_3) \leq N \\ p_j = (N + 2 - a_2)(N + 2 - a_3) \end{gathered}$$

From these relations, we get $p_i = 10,7,2$ and $p_j = 9,6,3$ and 8,3. Therefore, p_j= 3. (p_i,p_j)= (10,3),(7,3) when $p_i > p_j$. Also, $p_j = (8,3)$ when $p_i < p_j$. Therefore, (p_i,p_j) = (7,8), (2,8), (2,3). Only (7,8) and (2,3) are valid sets. Thus,

12;(1,4,9)$_1$: 1,2,3,4,5,6,7,11,12,8,9,10 and 12;(1,4,9)$_2$: 1,2,11,12,3,4,5,6,7,8,9,10.

Similarly,
(p_i, p_j) = (9,10),(4,5)

14;(1,4,9)$_1$: 1,2,11,12,3,4,5,6,7,8,9,13,14,10, 14;(1,4,9)$_2$: 1,2,11,12,3,4,13,14,5,6,7,8,9,10
14;(1,4,9)$_3$: 1,2,3,4,5,6,7,11,12,8,9,13,14,10, 14;(1,4,9)$_4$: 1,2,3,4,13,14,5,6,7,11,12,8,9,10

16;(1,4,9)$_1$: 1,2,11,12,3,4,5,6,15,16,7,8,9,13,14,10, 16;(1,4,9)$_2$: 1,2,11,15,16,12,3,4,5,6,7,8,9,13,14,10
16;(1,4,9)$_3$: 1,2,11,15,16,12,3,4,13,14,5,6,7,8,9,10, 16;(1,4,9)$_4$: 1,2,11,12,3,4,13,14,5,6,15,16,7,8,9,10
16;(1,4,9)$_5$: 1,2,3,4,5,6,7,11,15,16,12,8,9,13,14,10, 16;(1,4,9)$_6$: 1,2,3,4,5,6,15,16,7,11,12,8,9,13,14,10

16;(1,4,9)$_7$: 1,2,3,4,13,14,5,6,7,11,15,16,12,8,9,10, 16;(1,4,9)$_8$: 1,2,3,4,13,14,5,6,15,16,7,11,12,8,9,10
18;(1,4,9)$_1$: 1,2,11,12,3,4,5,6,15,16,7,8,17,18,9,10

To save the space other possible 15 necklaces are not listed.

20;(1,4,9)$_1$ = 1,2,11,12,3,4,5,6,15,19,20,16,7,8,17,18,9,10. Remaining 15 are not listed.

Thus, we can find all the possible necklaces for any even *N*.

Values of *N* were restricted because we started with some specific DNs. However, if we start with some other DN, we can still derive the DNs of higher lengths. This is explained with the following example.

Example 6: Let 8;(1,4): 1,2,3,4,8,7,6,5. Then from Equation (25), $p_i = 8{,}5$ and from Equations (26), (27), 7,4,1 and 6. There is no common value of p_j. From Equation (29), p_j = 6. Therefore, valid (p_i, p_j) = (5,6). Hence, 10;(1,4): 1,2,3,4,8,7,6,10,9,5 and 12;(1,4): 1,2,3,4,8,12,11,7,6,10,9,5, and so on.

Additional DFs

We observe that *N*+1 and *N*+2 are merged between two suitable numbers. Alternatively, we can increase all the numbers by 2 and then merge 1 and 2 suitably. Thus, we get the additional DFs. New value of 8;(1,4): 3,4,5,6,10,9,8,7. Merging 1 and 2 between 5 and 6, we get 10;(1,4): 3,4,5,1,2,6,10,9,8,7. Similarly, 12;(1,4): 5,1,2,6,7,3,4,8,12,11,10,9.

3.3 *Technique 3: A necklace merged into another necklace*

Consider the merging of DN *N*;(a_1, a_2) into another DN *n;*(a_1, a_2) shown in Figure 1(a) where the latter necklace has all the numbers of *N;*(a_1, a_2) increased by n. This will be referred as $\widetilde{N}; (a_1, a_2)$.

Without any loss of generality, let us assume

$$a_2 > a_1, p_j > p_i, n \geq N. \tag{30}$$

The two necklaces should satisfy

$$\begin{aligned} p_j - p_i &= (a_1, a_2) \\ \rightarrow p_j &= (p_i + a_1), (p_i + a_2). \end{aligned} \tag{31}$$

$$\text{and } q_g - p_i = (a_1, a_2) \tag{32}$$

Subtracting Equation (31) from Equation (32)

$$\begin{aligned} q_g - p_j &= (a_2 - a_1) \\ \rightarrow q_g &= p_j + (a_2 - a_1). \end{aligned} \tag{33}$$

Substituting for p_j from Equation (31),

$$\begin{aligned} q_g &= [(p_i + a_1), (p_i + a_2)] + (a_2 - a_1) \\ &\rightarrow [(p_i + a_2), (p_i + 2a_2 - a_1)]. \end{aligned} \tag{34}$$

Now $2a_2 - a_1$ has to be either a_1 or a_2. This is possible only when $a_1 = a_2$ which is not permissible as $a_1 < a_2$. Therefore,

$$q_g = (p_i + a_2). \tag{35}$$

From Equation (33),

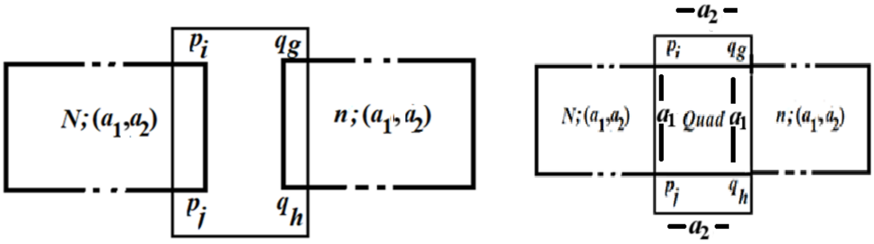
$$p_j = (p_i + a_1). \tag{36}$$

Also,

$$q_h = (q_g + a_1) = (p_i + a_2 + a_1). \tag{37}$$

Thus, Equations (35), (36) and (37) give q_g, p_j and q_h in terms of p_i.

If $n \leq N$, then interchange the roles of *N*;(a_1, a_2) and *n*;(a_1, a_2).



(a) (b)

Figure 1. (a) Two necklaces of lengths *N* and *n,* (b) Two necklaces merged together.

Procedure

Identify the pairs (p_i, p_j) in N;(a_1, a_2) which satisfy Equation (35). Then find possible q_g and q_h from Equations (35) and (37), respectively. Select the values of q_g and q_h greater than N, and also available side by side in $\widetilde{N}$;(a_1, a_2).

Equations (34)-(36) are pictorially shown as in Figure 2. This will be referred to a *quad* (p_i, q_g, q_h, p_j).

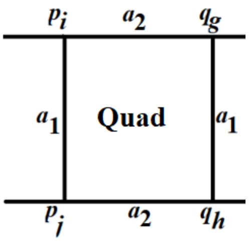
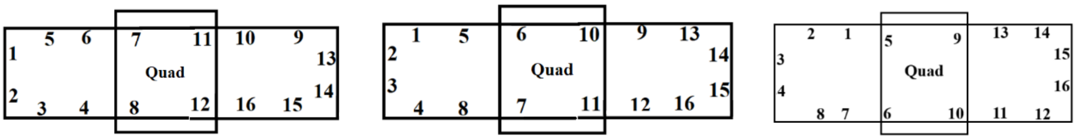


Figure 2. Quad (p_i, q_g, q_h, p_j) representation.

A) Two necklaces of the same length

Example 7: Find a necklace 16;(1,4) using 8;(1,4) = 1,2,3,4,8,7,6,5. Increasing the values by n = 8, we get $\widetilde{N}$;(1,4) = 9,10,11,12,16,15,14,13. Valid $(p_i, p_j) = (7,8), (6,7), (5,6)$. Corresponding (q_g, q_h) greater than 8 are (11,12), (10,11),(9,10). Therefore, the valid quads are (7,11,12,8), (6,10,11,7), (5,9,10,6). Thus, there are 3 possible DNs 16;(1,4) shown in Figure 3.



(a) (b) (c)

Figure 3. Three necklaces 16;(1,4).

Example 8: Find 58;(3.20) using 29;(3,20) = 1,4,24,27,7,10,13,16,19,22,2,5,25,28,8,11,14,17, 20,23,3,6,26,29, 9,12,15,18,21.

Increasing the numbers n = 29, we get $\widetilde{N}$;(3,20) = 30,33, 53, 56,36, 39,42,45,48,51,31,34,54,57,37,40,43, 46, 49, 52,32,35,55,58,38,41,44,47,50.

Valid quads: (25,45,48,28),(24,44,47,27),(10,30,33,13),(16,36,39,19),(19,39,42,22),(25,45,48,28), (11,31,34, 14). Thus, there are 7 DNs, only two of them are given below to save the space.

58;(3,20): corresponding to the quad (24,44,47,27) is
24,44,41,38,58,55,35,32,52,49,46,43,40,37,57,54,34,31,51,48,45,42,39,36,56,53,33,30,50,47,27,7,10,13,16,19,22,2,5,25,28,8,11,14,17,20,23,3,6,26,29,9,12,15,18,21.

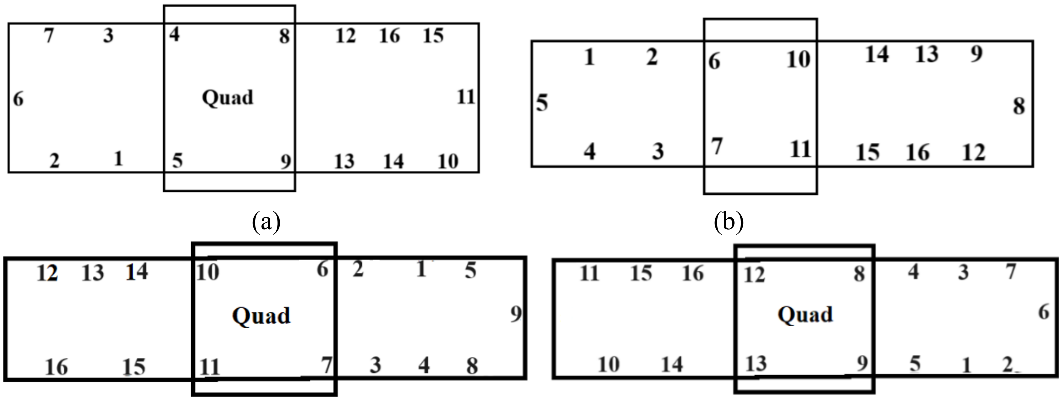
58;(3,20): corresponding to the quad (10,30,33,13) is
10,30,50,47,44,41,38,58,55,35,32,52,49,46,43,40,37,57,54,34,31,51,48,45,42,39,36,56,53,33,13,16,19,22,2,5,25,28,8,11,14,17,20,23,3,6,26,29,9,12,15,18,21,1,4,24,27,7.

Example 9: Let 10;(1,5): 1, 2, 7, 8, 3, 4, 9, 10, 5, 6.

Increasing the numbers by n = 10, we get $\widetilde{N}$;(1,5) = 11,12,17,18,13,14,19,20,15,16. Following the procedure, no valid quad is obtained. Hence two 20;(1,5) cannot be merged.

C) Two necklaces of different lengths

Example 9: Let 7;(1,4) = 1,2,6,7,3,4,5 be merged into 9;(1,4) = 1,2,6,7,3,4,8,9,5 to get DN 16;(1,4). Following the procedure, with n = 7, the valid quads are (4,8,9,5) and (6,7,10,11)). The corresponding DNs are shown in Figures 4(a) and (b). Instead of increasing the numbers in 9;(1,4) by 7, we increase the numbers in 7;(1,4) by 9, we get the additional DNs shown in Figures 4(c) and (d).



(a) (b)

(c) (d)

Figure 4. Four necklaces for 16;(1,4).

IV. APPLICATIONS

One application is to decorate the circular objects. Suppose a logo has 34 letters including the spaces. So, surround the logo with 34;(1,4) as shown in Figure 5. The 34;(1,4) is obtained by merging technique 3. First, merge 9;(1,4) = 8,9,5,1,2,6,7,3,4 and 8;(1,4) = 1,2,3,4,8,7,6,5 to get 17;(1,4) = 8,12,13,17,16,15,14,10,11, 12,13,9,5,1,2,6, 7,3,4. Then merge the two 17;(1,4) together.

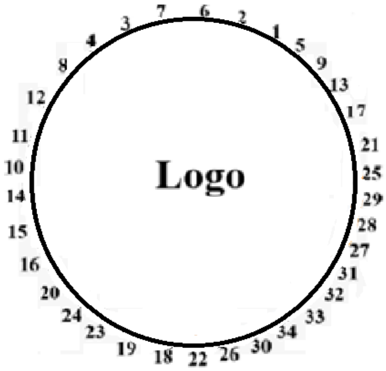


Figure 5. Decorated logo

Another application is in making a necklace of larger number of beads by merging necklaces of smaller ones.

V. CONCLUSION

Merging techniques for deriving the difference necklaces are presented. Technique 1 merges just one number, technique 2 merges two consecutive numbers; while the technique 3 merges one necklace into another necklace. Some of the statements made by White, Guy and Scheidler are modified to cover all possible cases. The number of necklaces increases as the length of the necklace increases. Some applications are given. We hope, readers will find interesting applications of this theory.

REFERENCES

[1]. Rathore, T. S. [2022] "Arranging integer numbers on a loop such that the sum of any two adjacent numbers is a perfect square, IEEE Region 10 Symposium, IIT Bombay, July 1-3.
[2]. Rathore, Tejmal [2022] "Chain-necklace diagrams with the sum of two consecutive numbers a perfect cube", AKGEC Int. J. of Technology, Vol. 13, No. 2, pp. 40-50.
[3]. Rathore, Tejmal [2022] "Chains/Necklaces for a wide range of algebraic operations on the two consecutive numbers restricted to a specified set of numbers", Research Inventy: Int. J. of Engineering and Science, Vol. 12, no. 8, pp. 01-04, 2022.
[4]. White, Ethan P. , Guy, Richard K. and Scheidler, Renate [2020] "Difference Necklaces", arXiv:2006.15250v1 [math.CO] 27 Jun 2020